Invariants puzzles

The Invariants

Long Vacation 2020

1 Week 1

1.1 Problem 1

1.1.1 Part I

There are n frogs of sizes $1, \dots, n$. In the ith second the frog of size i considers each of the remaining smaller frogs and eats each one independently with probability $\frac{1}{n}$ (it may not eat any frogs). What is the expected proportion of frogs remaining after n seconds, as n goes to infinity?

1.1.2 Part II

Now there are 25 frogs of sizes $1, \dots, 25$ sitting in a circle. At step 1, the frog of size 1 eats exactly one of the other frogs, each with equal probability. In the next step, the process repeats with the first surviving frog to the right of frog 1 eating exactly one of the other 23 remaining frogs. Generally, at step i for $2 \leq i \leq 24$, the first surviving frog to the right of the frog that ate during step $i-1$ will eat exactly one of the other $25-i$ remaining frogs, with equal probability. What is the probability that after 24 steps, frog 1 is the only remaining frog?

1.2 Problem 2

1.2.1 Part I

The evil giant Bonnor has 100 prisoners. He will give each prisoner a hat with a number from 1 to 100 (two hats can have the same number). Each prisoner can see the numbers on all other hats, but not the one on their own, and no communication of any form between the prisoners is allowed after the hats have been placed. All at once, the prisoners have to guess the number on their hat. If at least one prisoner gets their number right they will all go free, otherwise Bonnor will kill them all. If prisoners are allowed to talk beforehand to devise a strategy, is there a way for them to guarantee escape? If so, what is it?

1.2.2 Part II

An even more evil giant has also captured 100 prisoners, and each round he puts randomly numbered hats from 1 to 100 on the prisoners, as in part I. Again each prisoner guesses their own hat, and if no prisoner guesses correctly they all die. If at least two guess correctly, they are all set free. If exactly one guesses correctly however, they proceed to the next round and the entire process of placing and then guessing hats repeats with no chance in between for the prisoners to communicate. Assuming that every numbering of the hats is equally likely and that each numbering will come up after finitely many rounds (if the prisoners haven't escaped yet), can the prisoners come up with a strategy which will set them free after finitely many rounds?

1.3 Problem 3

1.3.1 Part I

A submarine travels secretly on the real line with a constant speed $c \in \mathbb{R}$, and it's speed and initial position are unknown to you. Every ith minute, for $i = 1, 2, \dots,$ after the departure of the submarine, you can fire a torpedo which will hit a closed interval of the real line of length exactly 1. Prove that there exists a strategy to fire the torpedoes so that you are guaranteed to hit the submarine after finitely many minutes (the time required to hit the submarine may depend on the initial position and speed of the submarine, but the strategy may not).

1.3.2 Part II

A submarine starts at the origin of the 2D plane and travels secretly with a constant speed and in a constant direction, both of which are unknown to you. Every ith minute, for $i = 1, 2, \dots,$ after the departure of the submarine, you can fire a torpedo which will hit a closed disk in the plane of diameter exactly 1. Is there a strategy to fire torpedoes so that you're guaranteed to hit the submarine after finitely many minutes.

2 Week 2

2.1 Problem 1

2.1.1 Part 1

We have a set of 100 positive integers with sum less than 200, prove that there is some subset of it whose sum is 100.

2.1.2 Part II

We again have a set of 100 positive integers with sum less than 200, is it necessarily the case that for any positive integer $k \leq 100$, there is a subset whose sum is k?

2.2 Problem 2

Mario and Wario play a game on a 1×100100 board, initially the all the squares of the board have a W written on them, each turn is as follows:

Mario goes first and chooses two chains of adjacent squares, then he turns every M in the chains into a W and every W in the chains into an M.

Then Wario chooses one chain of adjacent squares and turns every M in the chain into W and every W in the chain into an M.

Mario's goal is to maximize the number of M's at the end of the game while Wario's is to minimize it. Compute the maximum number of M's Mario can leave in the board after 500 turns if both of them play optimally.

2.3 Problem 3

2.3.1 Part I

Let n be a positive integer and $E_1, E_2, \cdots, E_n$ be non-empty subsets of $\{1, 2, \cdots, n\}$ so that there exist no disjoint non-empty sets $A, B \subseteq \{1, 2, \cdots, n\}$ with $\cup_{i \in A} E_i = \cup_{i \in B} E_i$. Prove that $\cup_{i=1}^{n} E_i = \{1, 2, \cdots, n\}$.

2.3.2 Part II

Suppose we are given uncountably many non-empty subsets $C_r \subseteq \mathbb{N}$, one for each $r \in \mathbb{R}$. Must there exist disjoint non-empty finite subsets $A, B \subseteq \mathbb{R}$ so that $\cup_{r \in A} C_r = \cup_{r \in B} C_r$?

3 Week 3

3.1 Problem 1

Let p be a prime number. Denote by $N(p)$ the smallest positive integer so that whenever we pick $N(p)$ (not necessarily distinct) row vectors $(a_i, b_i) \in \mathbb{N}^2$, there exist $i \neq j$ so that $\det \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix}$ is divisible by p. Find $N(p)$ for each prime p.

3.1.1 Solution

We claim that $N(p) = p + 2$. For the lower bound, it suffices to consider the $p+1$ row vectors $(1, 0)$ and $(i, 1)$ for $0 \leq i \leq p - 1$. Indeed, $\det \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = 1$ and $\det \begin{bmatrix} i & 1 \\ j & 1 \end{bmatrix} = i - j$ are not divisible by p. For the upper bound, we show that for any collection of $p + 2$ row vectors $(a_i, b_i) \in \mathbb{N}^2$, $1 \leq i \leq p + 2$, there exist $i \neq j$ so that $\det \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix}$ is divisible by p. First note that we can consider these row vectors as elements of $(\mathbb{Z}/p\mathbb{Z})^2$. Then clearly $b_k = 0$ for at most one value of k. Consider the remaining $p + 1$ vectors, which are wlog (a_i, b_i) for $1 \leq i \leq p + 1$. Then out of the $p + 1$ well-defined quotients $\frac{a_i}{b_i} \in \mathbb{Z}/p\mathbb{Z}$, at least two are equal mod p. Let l, r be such indices, then $\det \begin{bmatrix} a_l & b_l \\ a_r & b_r \end{bmatrix} \equiv 0$ as required.

3.2 Problem 2

There is a deck of 36 cards, with 9 of each suit. The cards get shuffled in a random order. Then a magicians assistant looks at them all and puts a yellow or blue dot on the back of each card according to some prearranged strategy. The magician then receives the deck and has to try to guess the suit of each card. He proceeds as follows: he looks at the back of the top card of the deck and announces what he thinks the suit of that card is, he then checks if he is correct and moves on to the next card. Can the magician and his assistant find a strategy that guarantees that once he has gone through all 36 cards, he can always:

a) Guess the suit of at least 19 of them?

b) Guess the suit of at least 20 of them?

3.3 Problem 3

Let m, n be positive integers and consider the unit interval $I = [0, 1]$. A cut consists of selecting exactly one of the current subintervals of I and partitioning it in exactly two subintervals. What is the least number of cuts needed so that out of the obtained subintervals one can make m segments of length $1/m$ each, and one can also make n segments of length $1/n$ each?

3.3.1 Solution when m, n are coprime.

If m, n are coprime, consider the n intervals $I_1, I_2, \ldots, I_n$ of length $1/n$ and m intervals $J_1, J_2, \ldots, J_m$ of length $1/m$. Suppose k is the least number of cuts required, then equivalently $k+1$ is the least number of 'pieces' (meaning intervals) into which we divided I. Consider now a bipartite graph G with vertex sets I_l, $1 \leq l \leq n$, and J_r, $1 \leq r \leq m$ and draw an edge between vertices I_l and J_r exactly when (at least) one of the 'pieces' used to fill I_l when dividing the pieces over the I-intervals is also used to fill J_r when dividing the pieces over the J-intervals. Then the number of edges in G is less than or equal to $k+1=$ 'number of pieces' (as each edge comes from at least one piece). To conclude we show that G needs to be connected so by standard results, as G has $m+n$ vertices, it needs at least $m+n-1$ edges, so $k \geq m+n-2$ (this is clearly also sufficient by cutting at the trivial places i/n for $1 \leq i \leq n-1$ and j/m for $1 \leq j \leq m-1$). Let H be a connected component of G, then it contains some number t of the I-intervals as vertices and some number s of the J-intervals as vertices, and all edges in H go between one of these t I-intervals and one of these s J-intervals. So exactly the same pieces are used for these t I-intervals as for these s J-intervals. Then the total length of the pieces is t/n but also s/m so $mt = sn$ and by coprimality, n divides t and m divides s so that H contains all vertices of G, i.e. G is connected. Answer: $k = n+m-2$.

4 Week 4

4.1 Problem 1

Let f be a bounded, measurable function on $[0, 1]$ with $\int_0^1 f(t)dt = 0$. Prove that for all $r \in (0, 1)$ there exists a subset J_r of $[0, 1]$ of measure r with $\int_{J_r} f(t)dt = 0$.

4.2 Problem 2

Let G be a simple graph. Prove that there exists a subset $V' \subseteq V(G)$ so that every vertex $v \in V'$ has an even number of neighbours in V', and every vertex $w \in V - V'$ has an odd number of neighbours in V'.

4.3 Problem 3

4.3.1 Part I

For an odd integer N, we have an $N \times N$ board covered in domino tiles except for one corner. We can slide a domino piece if the square we slide it to is empty. Prove that by a series of such moves we can move the empty square to any of the other corners.

4.3.2 Part II

With the same set-up as in Part I for a fixed odd integer N, which squares can the empty square always be moved to, regardless of the initial arrangement of dominoes?

5 Week 5

5.1 Problem 1

There are 30 rays coming out of the origin. For each pair of rays consider the smaller of the two angles between them (the one smaller than or equal to 180 degrees). What is the minimum possible number of acute angles amongst these?

5.2 Problem 2

Let $\{A_\alpha\}$ be a collection of distinct subsets of $\mathbb{Z}$ such that $A_\alpha \cap A_\beta$ is finite whenever $\alpha \neq \beta$. Prove that there exists such a collection whose cardinality is that of the continuum. If further N is a positive integer and we require that $|A_\alpha \cap A_\beta| \leq N$ whenever $\alpha \neq \beta$, does there still exist such a collection whose cardinality is that of the continuum?

5.3 Problem 3

Given a positive integer n and a positive constant C, prove that there exists an $n \times n$ matrix A with real entries such that A itself, and every matrix we can get from A by perturbing some (or all) of its entries by at most C, is invertible.

6 Week 6

6.1 Problem 1

One deals out a deck of 52 cards, faced up, into a 4×13 array. Then one tries to select 13 cards, one from each column, in such a way as to get one card of each denomination (but not necessarily of the same suit.) Must this always be possible?

6.2 Problem 2

Calculate the remainder when

$$\sum_{x=1}^{2016} \prod_{n=0}^{20} (1 + x^{2^n})$$

is divided by 2017.

6.3 Problem 3

Let $ABCD$ be a quadrilateral with $AB = 9, BC = 6, CD = 8, DA = 12$ and $AC = 10$. Let P be the point on BD such that $\angle APB = \angle BPC$. Find $\frac{DP}{PB}$.

7 Week 7

7.1 Problem 1

7.1.1 Part I

Is there a way to label the faces of a cube with a positive real number so that the number on each face equals the product of the numbers on the adjacent faces (i.e. the faces with which it shares an edge), other than the labelling with only ones?

7.1.2 Part II

Same question, but for the faces of an octahedron.

7.2 Problem 2

Let $m_1, m_2, \ldots, m_N$ be N integers. Show that

$$\prod_{1 \leq j < i \leq N} \frac{m_i - m_j}{i - j}$$

is also an integer.

7.3 Problem 3

Brian the consultant is standing in the centre of a square and in each corner of the square there is a wolf. Brian is fit so he can overpower one wolf, but he gets eaten if he fights two or more at the same time. The wolves' speed is 1.5 times that of Brian. Can Brian use his consulting skills to escape the square or are they useless leaving him to either starve or be eaten by wolves?

8 Week 8

8.1 Problem 1

For which positive natural numbers m does the inequality $|\sin(mx) - \sin(my)| \leq m|\sin(x-y)|$ hold for all real x, y?

8.2 Problem 2

Prove the following identity for integer $p \geq n+1$:

$$\sum_{l=0}^{n} (-1)^l \binom{p}{n-l} \binom{p-n+l-1}{l} = 1.$$

8.3 Problem 3

Silvia cuts strips out of a cake (the cake is a unit circle and the strips are regions bounded by parallel lines). Before she knows it Silvia ate the whole cake! Prove that the sum of the widths of the strips is at least 2.