Invariants puzzles

The Invariants

11$^{\text{th}}$ October, Michaelmas 2020

Submitting your solutions

Solutions should be submitted to puzzles@invariants.org.uk no later than 14:59 BST on Sunday 9$^{\text{th}}$ August. Please quote your name, your institution, the week, and the numbers of the problems attempted in the subject line, e.g. A.Example - University of Example - Week 1 - Problems 1,2,3.

Week 1

Problem 1

Let $S \subseteq \mathbb{N}$ such that $\sum_{n \in S} \frac{1}{n} < \infty$. Prove that

$$\lim_{n \to \infty} \frac{|S \cap \{1, \ldots, n\}|}{n} = 0.$$

Problem 2

Compute

$$\lim_{n \to \infty} \int_0^\pi \frac{\sin(x)}{1 + \cos^2(nx)} dx.$$

Problem 3

Find all solid convex bodies in three-dimensional space such that the projection onto any plane is a circle.