A15293W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part A: Paper A0
Honour School of Mathematics and Computer Science Part A: Paper A0
Honour School of Mathematics and Philosophy Part A: Paper A0
Honour School of Mathematics and Statistics Part A: Paper A0

Linear Algebra

TRINITY TERM 2016
WEDNESDAY, 15 JUNE 2016, 9.30am to 11.00am

- *Answers to the best two questions will count towards the total mark for the paper.*
- *All questions are worth 25 marks.*
- *You may hand in attempts to any number of questions.*
- *Begin the answer to each question in a new answer booklet.*
- *Hand in your answers in numerical order.*
- *Indicate on the front sheet the numbers of the questions attempted.*
- *A booklet with the front cover sheet completed must be handed in even if no question has been attempted.*
- *Cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such answer booklet and attach these answer booklets at the back of your work.*

Do not turn this page until you are told that you may do so

1. (a) [5 marks] Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, and that $T : V \to V$ is a linear transformation.

   (i) Prove that there exists a non-zero polynomial $p(x)$ such that $p(T) = 0$.

   (ii) Prove that there exists a unique monic polynomial $m_T(x)$ such that for all polynomials $q(x)$, $q(T) = 0$ if and only if $m_T(x)$ divides $q(x)$.

   (iii) State a criterion for diagonalisability of T in terms of $m_T(x)$.

   (b) [10 marks] Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$ and that $T : V \to V$ is a linear transformation.

   (i) Prove that for all $i \geqslant 0$, $\ker T^i$ is a subspace of $\ker T^{i+1}$.
   Let $B_1 \subseteq B_2 \subseteq \cdots$ be sets such that B_i is a basis for $\ker T^i$.

   (ii) Prove that if for some k, $T^k = 0$, then T is upper-triangularisable. Deduce that for any $\lambda \in \mathbb{F}$, if $(T - \lambda I)^k = 0$, then T is upper-triangularisable.

   (iii) Show that T is upper-triangularisable if and only if $m_T(x)$ is a product of linear factors.

   [*You may use the Primary Decomposition Theorem.*]

   (c) [10 marks] For which values of α and β is the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ \alpha - 1 & \alpha - \beta & \beta \\ \alpha - 1 & \alpha - \beta - 1 & \beta + 1 \end{pmatrix}$$

   diagonalisable over $\mathbb{R}$?

   For which values of α and β is it upper-triangularisable over $\mathbb{R}$?

2. (a) [15 marks] Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Suppose that $B = \{e_1, \ldots, e_n\}$ is a basis for V.

   (i) Define the *dual space* V' of V and the *dual basis* $B' = \{e'_1, \ldots, e'_n\}$. Prove that B' is indeed a basis for V'.

   (ii) If $T : V \to V$ is a linear transformation, define the *dual map* T'. State and prove a relationship between the matrices of T and T' with respect to the bases given. How are the characteristic polynomials of T and T' related? How are the minimum polynomials related? Justify your answers briefly.

   (iii) If U is a subspace of V, define the *annihilator* U° of U.

   (iv) Define a natural isomorphism Φ between V and its double dual V''. [*You do not need to give proofs that* Φ *is well-defined or that it is an isomorphism.*] Prove that if U is a subspace of V, then $\Phi|_U$ is a bijection between U and $U^{\circ\circ}$.

(b) [10 marks] Let V be the vector space of all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all but finitely many n, $f(n) = 0$, equipped with operations of vector addition and scalar multiplication defined so that $(f + g)(n) = f(n) + g(n)$ and $(\alpha f)(n) = \alpha f(n)$ for all $f, g \in V$, $n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$.

Define W to be the vector space of all functions from $\mathbb{N}$ to $\mathbb{R}$, with similarly defined operations of vector addition and scalar multiplication.

If $f \in W$, define $\theta_f : V \to \mathbb{R}$ so that

$$\theta_f(g) = \sum_{n=0}^{\infty} f(n)g(n).$$

Prove that the map $f \mapsto \theta_f$ is an isomorphism between W and V'.

Prove that the map $\Phi : V \to V''$ defined as in part (a) is not a surjection.

[*You may assume that if* U *is a vector space over* $\mathbb{R}$, L *is a linearly independent subset of* U, *and* $h : L \to \mathbb{R}$, *then there exists a linear functional* $k : U \to \mathbb{R}$ *such that* $k|_L = h$.]

3. Let V be a finite-dimensional inner-product space over $\mathbb{C}$.

   (a) [6 marks] Suppose that $T : V \to V$ is a linear transformation. Define the *adjoint map* T^*. (You do not need to prove that it exists or is unique.)

   Suppose that T has the property that $T^* = \alpha T$ for some $\alpha \in \mathbb{C}$. Prove that T is diagonalisable.

   (b) [9 marks] We say that T is *self-adjoint* if $T^* = T$, and that it is *skew-adjoint* if $T^* = -T$. Observe that if S and T are self-adjoint, then so are $S+T$, $S-T$, and βT, for any real number β.

   Recall that if $T : V \to V$ is any linear transformation, then $T + T^*$ is self-adjoint.

   (i) Prove that any linear transformation T can be written as the sum of a self-adjoint and a skew-adjoint linear transformation.
   Is it the case that a sum of diagonalisable linear transformations is diagonalisable? Give a proof or a counterexample.

   (ii) What are the possible eigenvalues of a self-adjoint linear transformation? Justify your answer carefully.

   (iii) Characterise the possible Jordan Normal Forms of linear transformations $T : V \to V$ such that T^2 is self-adjoint.

   (c) [10 marks] Suppose now that $T : V \to V$ is a linear transformation, and that $TT^* = T^*T$.

   (i) Prove that if v is an eigenvector of T^*, then $\langle v \rangle^\perp$ is T-invariant.

   (ii) Prove that if $V_\lambda = \ker(T - \lambda I)$, and $v \in V_\lambda$, then $T^*v \in V_\lambda$ also.

   (iii) Hence prove that there exists an orthogonal basis for V consisting of vectors which are eigenvectors for both T and T^*.

   (iv) Does it follow that T is self-adjoint? Give a proof or a counterexample.