Math Competition 2023, Fall Edition

Invariants Committee (Oxford) - BMX - The Archimedeans (Cambridge)

Nov 24^{th} 2023

Here are the 4 problems that you have to solve as part of the first day of the BMX Math Competition. We accept solutions send up until 25^{th} November, 11:59 PM CET time. For any questions regarding the problem do not hesitate to get in touch with us! Good luck to all of you!

1 Exam Questions, Day 1

Problem 1

A group of $2n+1$ individuals follows the property: for any group of n persons, there exists one person among the remaining $n+1$ such that he knows each of the n individuals. Prove that there exists a person who knows all the other $2n$ individuals.
Observation: A knows B implies that B knows A as well.

Problem 2

Anna and Bob play the following game. Initially, there are 2000 points in the plane, any two of which are connected by an edge. Bob can delete one edge on his round, and Anna can delete two or three edges on her round. The player that cannot make any more deletion without reaching a state with a point with no edges (an isolated point), loses. Bob moves first. Who has a winning strategy?

Bonus: Solve the problem for any general number n of points in the plane (this is purely for your enrichment, it will not yield bonus points)

Problem 3

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the condition:

$$f([x]\{y\}) = [f(x)]\{f(y)\} \quad .$$

for all $x, y \in \mathbb{R}$.

Here, we denoted by $[x]$ the floor function of x, meaning that $[x]$ is the greatest integer smaller or equal than x, and $\{x\}$, the fractional part of x, which is defined as $\{x\} = x - [x]$.

Problem 4

Let X be an uncountable subset of $[0, 1]$ (that is, there is no injective function $f : X \to \mathbb{N}$). Prove the following:

1. There exists $x \in X$ such that, for any $\varepsilon > 0$ we have that $(x - \varepsilon, x + \varepsilon) \cap X$ is uncountable.
2. There exists Y a subset of X such that, for any $y \in Y$ and any $\varepsilon > 0$ we have that $(y-\varepsilon, y+\varepsilon) \cap Y$ is uncountable.

(In order to solve question 2, you can assume that the result from question 1 is true.)