puzzles & pizza

november 2024

red flag!

This question has a bounty! Send solutions to `puzzles@invariants.org.uk` by 25^{th} November (Monday of Week 7) at 4pm, UK time.

Consider a $2n \times 2n$ lattice points grid

$$S = \{(x, y) \in \mathbb{Z}^2 : 0 \leq x, y \leq 2n - 1\}$$

for some integer $n \geq 2$. At the point $(0, n - 1)$ we place a white flag while the remaining $4n^2 - 1$ points have a red flag.

At each step, we can switch the colour of all the flags from any row, column, or a parallel to any of the diagonals. Can we have all red flags after a finite number of iterations?

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the perfect riffle

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A perfect riffle on a sequence of length 2^n is defined as follows: split the sequence in half, and interleave the two sequences together. For example, a perfect riffle on the sequence abcdefgh gives aebfcgdh.

Consider a bit integer $b_1 b_2 \ldots b_{2^k}$, where each b_i is either 0 or 1, which represents some integer B in binary (i.e., $B = 2^{2^k-1}b_1 + 2^{2^k-2}b_2 + \cdots + b_{2^k}$). By applying a perfect riffle on the b_i's we get a bijection

$$r : \{0, 1, \ldots, 2^{2^k} - 1\} \to \{0, 1, \ldots, 2^{2^k} - 1\}$$

Now let $\text{minRiffle}(B) = \min\{B, r(B), r^2(B), \ldots\}$. For each k, find a 2^k bit integer B that maximises $B/\text{minRiffle}(B)$.

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probability and pagers

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Take a binary expansion $X = 0.x_1x_2x_3\ldots$ where, for each $i \in \mathbb{N}$, $\mathbb{P}(x_i = 1) = 2/3$ and $\mathbb{P}(x_i = 0) = 1/3$.

(a) Using careful reasoning to justify your answers, what are the values of $\mathbb{P}(X > 1/3)$, $\mathbb{E}[X]$ and $\mathbb{P}(X \in \mathbb{Q})$?

(b) X is sent via a pager, and is displayed as the binary expansion $Y = 0.y_1y_2y_3\ldots$. The pager is *catastrophically lossy*, in the following way: for some $u, k \in (0,1)$, $\mathbb{P}(y_1 = x_1) = u$, and for each $i \in \mathbb{N}$,

$$\mathbb{P}(y_{i+1} = x_{i+1} \mid y_i = x_i) = \mathbb{P}(y_i = x_i)$$
$$\mathbb{P}(y_{i+1} = x_{i+1} \mid y_i \neq x_i) = \max\{0, \mathbb{P}(y_i = x_i) - k\}$$
$$\mathbb{P}(y_i = x_i \mid x_i = 1) = \mathbb{P}(y_i = x_i \mid x_i = 0)$$

What is $\mathbb{P}(y_i = 1)$, for each $i \in \mathbb{N}$?

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liars

The following people, $P_1, \ldots, P_{2^{99}}$, say

$$\begin{aligned} P_1 &: \text{My number is odd.} \\ P_2 &: P_1 \text{ is lying.} \\ &\vdots \\ P_{2k-1} &: \text{My number is odd.} \\ P_{2k} &: P_k \text{ is lying.} \\ &\vdots \\ P_{2^{99}-1} &: \text{My number is odd.} \\ P_{2^{99}} &: P_{2^{98}} \text{ is lying.} \end{aligned}$$

How many people are lying?

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keep your distance

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism.

(a) Suppose there exists $x, y \in \mathbb{R}^n$ such that $x \neq y$ and $|T(x) - T(y)| = |x - y|$. Need T be an isometry?

(b) Suppose there exists $c \in \mathbb{R}^n$ such that, for any $x \in \mathbb{R}^n$ we have $|T(c) - T(x)| = |c - x|$. Need T be an isometry?

(c) Suppose for any $x, y, z \in \mathbb{R}^n$, at least one of the following equations is true:

- $|T(x) - T(y)| = |x - y|$;
- $|T(x) - T(z)| = |x - z|$;
- $|T(y) - T(z)| = |y - z|$.

Need T be an isometry?