Oxford Mathematics Team Challenge
Guts Round Solutions (with Questions)

Saturday, 8th March 2025

At the start of the next page are solutions to each question of the Guts Round. In the last three questions (the last set) we only provide the actual values.

Errata

There were no errors in this year's Guts Round.

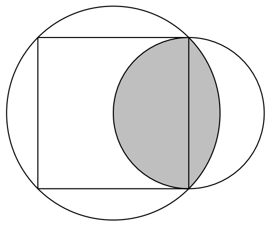
1. [3 points] Two countries each have an Olympic tennis squad of ten players. Each country chooses two of their players to compete in a doubles match. How many possible lineups are there for the match?

Sol. Each country has ${}^{10}C_2$ ways to select 2 players to compete in the match. Therefore the number of possible lineups is $({}^{10}C_2)^2 = \boxed{2025}$.

2. [3 points] What is the third smallest positive integer whose digits sum to 25?

Sol. The first few positive integers whose digits sum to 25 is 799, 889, and 898. Therefore the answer is $\boxed{898}$.

3. [3 points] Jimbo is drawing a coffee bean. He first draws a square of side length 2, draws a circle whose diameter is a side of the square, and then draws another circle circumscribing the square. The intersecting region of the two circles is his coffee bean. What is the area of his coffee bean?



Sol. First, we note that the radius of the bigger circle has a radius of $\sqrt{2}$. To calculate the area of the coffee bean, we split the shaded area into two regions. The first region, given by the area of half the small circle, has an area of $\frac{1}{2}\pi$. The second region has $\frac{1}{4}$ the area of the bigger circle minus the area of the square giving it an area of $\frac{1}{2}\pi - 1$. Thus, the total area of his coffee bean is $\boxed{\pi - 1}$.

4. [4 points] Mr. Cheng needs to split his class of 7 students into 2 groups for a project, one group with 3 students and another with 4 students. Two of Mr. Cheng's students are Richard and Alice, who must be kept in separate groups because they quite strongly dislike each other. How many ways can Mr. Cheng form the two groups?

Sol. Assuming that Alice and Richard are in two separate groups, the question essentially boils down to Mr. Cheng splitting the rest of his class (of 5 students) into one group with 2 students and another with 3 students. But as Alice and Richard could switch groups we need to multiply the final answer by 2 giving the total to be $2 \cdot {}^5C_2 = \boxed{20}$.

5. [4 points] The rectangle $ABCD$ has side lengths $AB = 9$, $BC = 6$. Point P is chosen inside the rectangle such that the areas of $ABPD$, BCP and CDP are all equal. Find the length of AP.

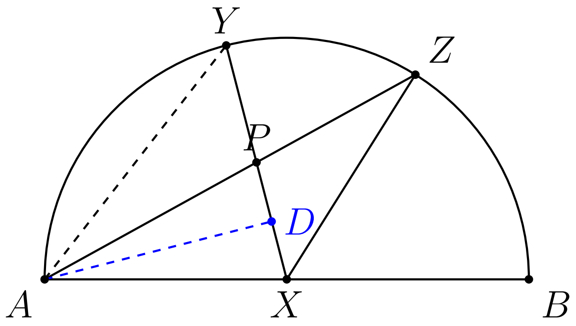
Sol. As the areas of BCP and CDP are equal, we know P must lie on the diagonal AC. Now we also know the ratio of areas $ABPD : BCP + CDP = AP : PC = 1 : 2$. And thus, the length of $AP = \frac{1}{3}\sqrt{6^2 + 9^2} = \boxed{\sqrt{13}}$.

6. [4 points] The positive integer N satisfies $108^2 + 144^2 = 2025N^2$. What is the value of N?

Sol. We can rewrite the left side of the equation to $(36 \cdot 3)^2 + (36 \cdot 4)^2 = 36^2 \cdot 3^2 + 36^2 \cdot 4^2 = 36^2 \cdot (3^2 + 4^2) = 36^2 \cdot 5^2 = (36 \cdot 5)^2 = 180^2$. We can rewrite the right side of the equation to $45^2 \cdot N^2 = (45 \cdot N)^2$. So $180 = 45 \cdot N$, so $N = \boxed{4}$.

7. [5 points] Let AB be the diameter of a semicircle, and let X lie on line segment AB such that $AX = BX$. Choose points Y and Z on the arc of the semicircle such that $AY < AZ$. Let P be the intersection of AZ and XY, and suppose that $AP = 4$, $PX = 2$, $PY = 2$, and $PZ = 3$. Find AY.

Sol. We know that $YX = 4$. But as X is the midpoint of AB which is the diameter of the semicircle, we know that X is indeed the centre of the semicircle. Thus $XA = XY = XB = 4$. Now $XY = XA = 4 = AP$, so XAP is an isosceles triangle. Drawing a perpendicular from A to PX and calling this point D, we know that AP is perpendicular to PX, D bisects PX and thus $DX = 1$. So by Pythagoras $AD = \sqrt{4^2 - 1^2} = \sqrt{15}$ and $AY = \sqrt{AD^2 + YD^2} = \sqrt{15 + 9} = \sqrt{24} = \boxed{2\sqrt{6}}$.



8. [5 points] Noah has four playing cards with the words NO, OH, AN, and HA written on their faces, each card having a different word. He randomly arranges the four cards face-down into a row and then flips them over revealing the words on them. What is the probability that the first card's word contains the letter N, the second word contains O, the third word contains A, and that the fourth word contains H? Express your answer as a common fraction.

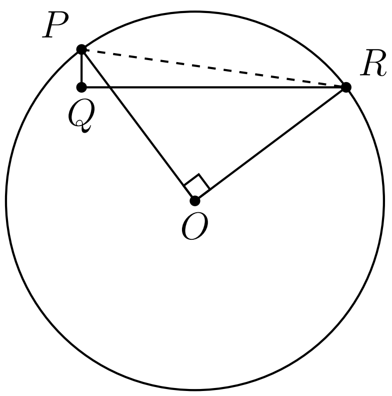
Sol. The total number of ways the four playing cards could be arranged is $4! = 24$. Now there are only 2 ways where the first card's word contains the letter N, the second word contains O, the third word contains A, and the fourth word contains H. Namely $\boxed{\text{NO}}\boxed{\text{OH}}\boxed{\text{AN}}\boxed{\text{HA}}$ and $\boxed{\text{AN}}\boxed{\text{NO}}\boxed{\text{HA}}\boxed{\text{OH}}$. Thus the probability is $\frac{2}{24} = \boxed{\frac{1}{12}}$.

9. [5 points] Let S be the set containing the terms in the arithmetic sequence $1, 8, 15, \ldots$ that are less than 2000. Find the smallest n such that, for any n numbers picked from S, there always will be a pair of numbers picked that add up to 2025.

Sol. The n^{th} term of the arithmetic sequence is given by the formula $a_n = 1 + 7 \cdot (n - 1)$. Suppose we group the members of the arithmetic sequence as follows: $\{1\}$, $\{8\}$, $\{15\}$, $\{22\}$, $\{29,\ 1996\}$, $\{36,\ 1989\}$, $\ldots$, $\{1+7\cdot i, 2024-7\cdot i\}$, $\ldots$, $\{1009,\ 1016\}$. Note we constructed 145 groups. Now if there are two numbers in the same group within the n numbers we picked from S, there will necessarily be a pair of numbers that add up to 2025. So (by the pigeonhole principle) the smallest n we can pick to guarantee there to be a pair of numbers that add to 2025 is $\boxed{146}$.

10. [5 points] Let P and R be two points on a circle of radius 5. Point Q is inside the circle. Suppose that $PQ = 1$, $QR = 7$, and $\angle PQR = 90°$. The segments PQ and QR together split the circle into two regions. What is the area of the smaller region?

Sol. $PR = \sqrt{50}$. If we denote the centre of the circle to be O, we know that $OP = OR = 5$, so by the converse of the Pythagorean theorem, $\angle POR = 90°$. After some gruelling calculations, we get that the area of the two regions is $\frac{25}{4}\pi - 9$ and $\frac{75}{4}\pi + 9$; the smaller area is $\boxed{\frac{25}{4}\pi - 9}$.



11. [5 points] Each morning, Arav has to decide how to get from his home to his maths class, which is 5 miles due south of his home. He can cycle at 10 mph; alternatively, he can jog at 6 mph to the bus stop a mile due north of his home, then take the bus which arrives every 5 minutes and takes a direct route to the maths class at 20 mph. If Arav leaves his house at a random time, what is the probability that the bus will get him to class faster than cycling?

Sol. Cycling to class takes 30 minutes. Jogging from his home to the bus stop takes 10 minutes. Taking the bus from the bus stop to his maths class takes 18 minutes. So for the bus to get him to class faster than cycling, Arav would need the bus to arrive within 2 minutes of him getting to the bus stop. As the bus arrives every 5 minutes, and Arav leaves his house at a random time, the probability is thus $\boxed{\frac{2}{5}}$.

12. [5 points] Jimbo wants to draw a triangle, but he chooses to be a bit chaotic. Every minute, he rolls a standard 6-sided die to help him decide how much of the triangle to draw. If he rolls a 6, he draws all remaining edges of the triangle. If he rolls a 4 or 5, he draws two more edges, unless two edges have already been drawn, in which case he draws just one more edge. If he rolls a 3 or lower, he draws one more edge. What is the probability that Jimbo takes exactly two minutes to draw the triangle?

Sol. There are 2 cases where Jimbo takes exactly two minutes to draw the triangle: Either he starts with (4,5) and ends with (1,2,3,4,5,6) or he starts with (1,2,3) and ends with (4,5,6). The probability is thus $\frac{(2\cdot6+3\cdot3)}{36} = \boxed{\frac{7}{12}}$.

13. [6 points] How many positive integers n have the property that $\frac{1}{n}$ is a recurring decimal with the shortest cycle being 3 digits long?

Sol. For $\frac{1}{n}$ to have a cycle of 3 digits, we know n is a factor of 999 as $\frac{1}{999} = 0.001001001\ldots$. The factors of 999 are 1, 3, 9, 27, 37, 111, 333, and 999. A quick check will tell us that only $n = 27$, 37, 111, 333, and 999 satisfy the property that $\frac{1}{n}$ has its shortest cycle being 3 digits long. Thus the answer is $\boxed{5}$.

14. [6 points] Find the denominator of

$$\frac{1(1+3)(1+3+5)\cdots(1+3+\cdots+23+25)}{20!}$$

when fully simplified.

Sol. Note $\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2n-1) = n^2$. So, the numerator is equal to $1^2 \times 2^2 \times \cdots \times 13^2$. So after some cancellations, what's left on the denominator will be the primes greater than 13. So the answer is $17 \times 19 = \boxed{323}$.

15. [6 points] Let $a_1, a_2, a_3, \ldots$ be a sequence of integers with the following properties:

- For every odd positive integer m, $a_m = m$.
- For every positive integer n, the sum of the terms from a_1 up to and including a_n equals the sum of the terms from a_{n+1} up to and including a_{2n}.

Find a_{100}.

Sol. As given in the question we know that $a_{2n-1} = 2n - 1$ and

$$a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n}$$

and in particular, we know

$$\begin{aligned} a_1 + \cdots + a_{50} &= a_{51} + \cdots + a_{100} \\ a_1 + \cdots + a_{49} &= a_{50} + \cdots + a_{98} \end{aligned}$$

Now,

$$\begin{aligned} a_{100} &= a_1 + \cdots + a_{50} - (a_{51} + \cdots + a_{99}) \\ &= a_1 + \cdots + a_{49} - (a_{50} + \cdots + a_{98}) + 2a_{50} - a_{99} \\ &= 2a_{50} - a_{99} \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} a_{50} &= 2a_{25} - 49 \\ &= 1 \end{aligned}$$

Thus, $a_{100} = \boxed{-97}$.

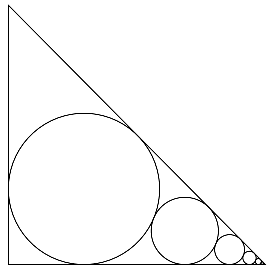
16. [6 points] Angelina has four outfits, and each morning, she chooses one of them to wear for the day. She cannot wear the same outfit on two consecutive days. On Sundays, Wednesdays, and Saturdays, she goes on a romantic dinner date with Eren, and so Angelina also cannot wear the same outfit to two consecutive romantic dinners. If Angelina wore her favourite outfit on Saturday, in how many different ways can she choose her outfits for the next week, from Sunday up to and including Saturday?

Sol. Count. Note the number of different ways Angelina can choose her outfits from Thursday to Saturday is the same number of ways she can choose her outfits from Monday to Wednesday (as each date's outfit only depends on the last date and we are working over the same time frame of 3 days). Suppose we labeled the outfits A, B, C, and D. There are 3 choices of outfits she can wear on Sunday. WLOG let her wear outfit A on Sunday. Then on Monday she can wear any of B, C, and D. If she wears outfit A on Tuesday, she can wear any of B, C, D on Wednesday for her date. But if she wears another outfit on Tuesday, then she only has 2 options to choose from on Wednesday (as she also cannot wear outfit A on Wednesday). Hence, the number of different ways she can choose her outfit from Monday to Wednesday is $3 \cdot (1 \cdot 3 + 2 \cdot 2) = 21$. Thus, the number of different ways she can choose her outfit from Sunday to Saturday is $3 \cdot 21 \cdot 21 = \boxed{1323}$.

17. [6 points] Let a and b be single-digit positive integers, such that $a > b$, and suppose that $a^4 - b^4$ has a total of 6 factors. What is the sum of all possible values of $a^4 - b^4$?

Sol. First note that $a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$, so that 1, $a - b$, $a + b$, $a^2 + b^2$, $(a + b)(a - b)$, $(a^2 + b^2)(a - b)$, $(a^2 + b^2)(a + b)$, $(a^2 + b^2)(a + b)(a - b) = a^4 - b^4$ are 8 numbers that divide $a^4 - b^4$, which has 6 factors so that these numbers can't all be distinct. As $0 < a - b < a + b < a^2 + b^2$, the only way for this to happen is for $a - b = 1$. This leaves us with 8 possible solutions. Checking each of these solutions, we find that (a, b) is either $(4, 3)$ or $(5, 4)$ and so our answer is $175 + 369 = \boxed{544}$.

18. [6 points] Jimbo stumbles upon the fractal shown below. The bottom side of the right triangle has length $\frac{17}{5}$, and the largest circle has radius 1. Assuming that infinitely many circles are drawn following this pattern, what is the sum of the areas of all the circles?



Sol. Suppose we denote the bottom right vertex of the right triangle as A and the bottom left vertex as B. Let O_1 be the centre of the largest circle with radius 1, and O_2 be the centre of the second largest circle with radius r. Let D and E be where the circle centred at O_1 touches AB and where the circle centred at O_2 touches AB respectively. Note O_1, O_2, and A are collinear. $AD = \frac{17}{5} - 1 = \frac{12}{5}$ so by Pythagoras $O_1A = \frac{13}{5}$. Hence, $O_2A = \frac{13}{5} - 1 - r = \frac{8}{5} - r$. But as O_1AD and O_2AE are similar triangles (by AAA) we must have that $\dfrac{O_1A}{O_1D} = \dfrac{O_2A}{O_2E}$, so plugging in the numbers we have $\frac{13}{5} = \frac{\frac{8}{5}-r}{r}$. Solving for r yields $r = \frac{4}{9}$. Now this is the ratio in radii between any two consecutive circles, and thus the total area of all the circles is given by $\pi + \frac{4}{9}\pi + \left(\frac{4}{9}\right)^2\pi + \ldots$. As this is the sum of a geometric series with ratio $\frac{16}{81}$ we get that the sum of the areas of all the circles is $\dfrac{\pi}{1-\frac{16}{81}} = \boxed{\dfrac{81}{65}\pi}$.

19. [7 points] Let $S = \{s_1, s_2, ..., s_n\}$ be the set of all positive factors of 2025. Find the value of

$$\frac{\log_{10}(s_1) + \log_{10}(s_2) + \cdots + \log_{10}(s_n)}{\log_{10}(2025)}.$$

Sol. By log rules

$$\frac{\log_{10}(s_1) + \log_{10}(s_2) + \cdots + \log_{10}(s_n)}{\log_{10}(2025)} = \frac{\log_{10}(s_1 s_2 \ldots s_n)}{\log_{10}(2025)}$$

Now $2025 = 3^4 \cdot 5^2$ so 2025 has $(4+1)(2+1) = 15$ factors. Each factor 'pairs' up with another factor to give a product of 2025 except for 45 as

$2025 = 45^2$. So $s_1 s_2 \ldots s_n = 2025^7 \cdot 45 = 45^{15}$. So

$$\frac{\log_{10}(s_1 s_2 \ldots s_n)}{\log_{10}(2025)} = \frac{\log_{10}(45^{15})}{\log_{10}(45^2)} = \frac{15\log_{10}(45)}{2\log_{10}(45)} = \boxed{\frac{15}{2}}.$$

20. [7 points] Aryan and Bao plan to meet each other on a 3 km long bridge. The bridge has automatic gates on the west and east entrances. The west-side gate opens briefly every 0, 20, and 40 minutes past the hour. The east-side gate opens briefly every 0 and 30 minutes past the hour. Aryan and Bao arrive at opposite ends of the bridge at exactly x minutes past noon, where x is an integer and $0 \leq x < 60$. As soon as the gate opens, each of them begins to walk along the bridge at 2 ms^{-1}. How many values of x allow Aryan and Bao to meet in less than 30 minutes of arriving at the bridge?

Sol. Aryan and Bao must walk a total of 3 km, which would take them $3000/2 = 1500$ seconds, equal to $1500/60 = 25$ minutes. So, the gates on both sides of the bridge must open fast enough so that they have more than 25 minutes between the both of them to walk across. Between the two of them, they have 60 minutes to spend, so that they can spend less than 35 minutes waiting for the gates to open. Let $W(x)$ be the number of minutes that the person on the west-side has to wait, and similarly let $E(x)$ be the number of minutes that the person on the east-side has to wait. The answer is the number of non-negative integers $x < 60$ such that $W(x)+E(x) < 35$. Using the fact that $W(x)$ is the remainder when x is divided by 20 and $E(x)$ is the remainder when x is divided by 30, we find that $W(x) + E(x) < 35$ exactly when x is one of $0, \ldots, 17$, $20, \ldots, 27$, $30, \ldots, 52$. Our answer is therefore $\boxed{49}$.

21. [7 points] Julian is ordering pizzas for a very large event. The pizza shop charges him 10 pounds per pizza, and the cost of delivery depends on the size of the order. An order of n pizzas costs $n + 1 - \lfloor\sqrt{n}\rfloor$ pounds to deliver, where $\lfloor\sqrt{n}\rfloor$ represents the largest integer less than or equal to $\sqrt{n}$. Julian must order at least 100 pizzas in total, but he can place multiple orders to be delivered separately. What is the minimum amount that Julian spends on pizzas, in pounds?

Sol. Consider the average price of delivery for n pizzas which is given by

$$f(n) = \frac{n+1-\lfloor\sqrt{n}\rfloor}{n} = 1 + \frac{1}{n} - \frac{\lfloor\sqrt{n}\rfloor}{n}$$

Setting $n = k^2 + m$ where $0 \le m < 2k+1$ we get

$$f(k^2+m) = 1 + \frac{1}{k^2+m} - \frac{k}{k^2+m} = 1 - \frac{k-1}{k^2+m} \ge 1 - \frac{k-1}{k^2}$$

Consider

$$\frac{k-1}{k^2} \ge \frac{k}{(k+1)^2}$$

This is true for $k \ge 2$. So we know for all $k \ge 2$ that $\frac{1}{4} \ge \frac{(k-1)}{k^2}$ with equality if and only if $k = 2$. Checking $k = 1$, we can see that $\frac{1-1}{1} \le \frac{1}{4}$, so indeed over the positive integers, the maximum of $\frac{k}{(k+1)^2}$ is $\frac{1}{4}$.

Thus, $1 - \frac{k-1}{k^2}$ achieves its minimum when $k = 2$, ie. when $n = 4$. So, the minimum amount that Julian spends on 100 pizzas will be when he buys 4 pizzas at a time, giving him a grand total of $100 \cdot 10 + \frac{3}{4} \cdot 100 = \boxed{1075}$.

22. [8 points] Bernie invents a function σ defined by

$$\sigma(s) = \sum_{n=2}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Find the value of

$$\sum_{s=2}^{\infty} \sigma(s) = \sigma(2) + \sigma(3) + \sigma(4) + \cdots$$

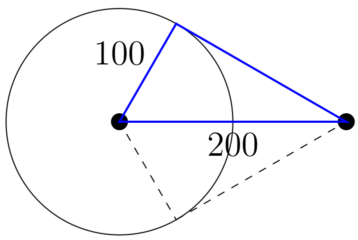
Sol. Swap the order of summation. Consider a fixed $n \ge 2$. Each term $\sigma(s)$ of the sum of Bernie's functions will contain $\frac{1}{n^s}$, and adding these up we find that these terms contribute

$$\begin{aligned}\sum_{s=2}^{\infty} \frac{1}{n^s} &= \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots \\ &= \frac{1/n^2}{1-1/n} \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.\end{aligned}$$

We can now look at the 'partial sums' of Bernie's functions, where we add up the contributions from each n. Starting with $n = 2$, we get $1 - \frac{1}{2}$. Adding in the contribution from $n = 3$ yields $1 - \frac{1}{3}$. Continuing in this way, we see that for any $k > 2$, the sum from $n = 2$ up to $n = k$ is $1 - \frac{1}{k}$. As k grows larger, $\frac{1}{k}$ approaches 0 and hence the sum of Bernie's functions approaches $\boxed{1}$.

23. [8 points] Sam and Owen are stood 200 m apart in a large foggy field. If they ever get within 100 m of each other, they will see each other. Sam and Owen both turn to a random direction and simultaneously begin walking in a straight path. Sam walks at a constant rate of 1 ms^{-1}, and Owen walks at a constant rate of 2 ms^{-1}. What is the probability that they will see each other?

Sol. Because Sam and Owen walk at constant rates, then from Sam's frame of reference, Owen is moving at a constant rate in a random direction. The rate at which Owen moves relative to Sam is unimportant, as we only need to know in what direction Owen needs to move at in order to eventually get within a distance of 100 m of Sam. By drawing a circle of radius 100 m around Sam, we find that the lines intersecting Owen's position that are tangent to the circle form a 60° angle, since the blue right triangle shown below has the side length ratio of a half equilateral triangle.



This means that in Sam's reference frame, as long as Owen moves in a direction within a 60° window, then he and Sam will see each other. This means the probability is $\dfrac{60}{360} = \boxed{\dfrac{1}{6}}$.

24. [8 points] Let AB be the diameter of a semicircle, and let X lie on line segment AB such that $AX \neq BX$. Choose points Y and Z on the arc of the semicircle such that $AY < AZ$. Let P be the intersection of AZ and XY, and suppose that $AP = 4$, $PX = 2$, $PY = 2$, and $PZ = 3$. Find AY.

Sol. Let us extend the semicircle into a full circle and let us call this circle ω. Suppose YX intersects ω at another point Q. Then by power of a point $YP \cdot PQ = AP \cdot PZ$. Hence, $PQ = 6$ and thus $XQ = 4$. But $XQ = 4 = XY$ and X is not the centre of the circle (as $AX \neq BX$), Thus, YQ must be perpendicular to AB. Applying Pythagoras, we get $AX = \sqrt{4^2 - 2^2} = \sqrt{12}$ and by Pythagoras again, $AY = \sqrt{\sqrt{12}^2 + 4^2} = \sqrt{28} = \boxed{2\sqrt{7}}$.

25. [16 points] Let N be the number of positive integer divisors of 2025!. How many digits does N have?

Ans. 192

26. [16 points] Ivy has a perfectly circular pizza and an atomically precise pizza cutter. Every second, she chooses two random points on the circumference of the pizza and then makes a perfectly straight cut connecting those two points. After she makes 2025 cuts, what is the expected number of pieces of pizza she makes? That is to say, if Ivy were to repeat this procedure sufficiently many times and take the mean of all her results, what number would this mean approach?

Ans. 685126

27. [16 points] What is the total number of points scored by all participants in today's OMTC on all questions, except this one?

Ans. Hopefully you remembered that each team's score on the OMTC 2025 is out of 700 points, and used the fact that there were 20 teams competing in-person (or 40 teams online). By thoroughly vibechecking the difficulty of the competition, you would find that the total number of points scored before the final question was $\boxed{3402.35}$ (in-person) or $\boxed{8508.6}$ (online).