Oxford Mathematics Team Challenge
Guts Round Solutions

Saturday, 8th March 2025

At the start of the next page are solutions to each question of the Guts Round. In the last three questions (the last set) we only provide the actual values.

Errata

There were no errors in this year's Guts Round.

1. [3 points] Each country has ${}^{10}C_2$ ways to select 2 players to compete in the match. Therefore the number of possible lineups is $({}^{10}C_2)^2 = \boxed{2025}$.

2. [3 points] The first few positive integers whose digits sum to 25 is 799, 889, and 898. Therefore the answer is $\boxed{898}$.

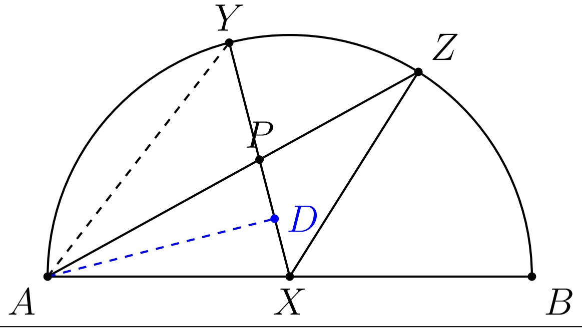
3. [3 points] First, we note that the radius of the bigger circle has a radius of $\sqrt{2}$. To calculate the area of the coffee bean, we split the shaded area into two regions. The first region, given by the area of half the small circle, has an area of $\frac{1}{2}\pi$. The second region has $\frac{1}{4}$ the area of the bigger circle minus the area of the square giving it an area of $\frac{1}{2}\pi - 1$. Thus, the total area of his coffee bean is $\boxed{\pi - 1}$.

4. [4 points] Assuming that Alice and Richard are in two separate groups, the question essentially boils down to Mr. Cheng splitting the rest of his class (of 5 students) into one group with 2 students and another with 3 students. But as Alice and Richard could switch groups we need to multiply the final answer by 2 giving the total to be $2 \cdot {}^5C_2 = \boxed{20}$.

5. [4 points] As the areas of BCP and CDP are equal, we know P must lie on the diagonal AC. Now we also know the ratio of areas $ABPD : BCP + CDP = AP : PC = 1 : 2$. And thus, the length of $AP = \frac{1}{3}\sqrt{6^2 + 9^2} = \boxed{\sqrt{13}}$.

6. [4 points] We can rewrite the left side of the equation to $(36 \cdot 3)^2 + (36 \cdot 4)^2 = 36^2 \cdot 3^2 + 36^2 \cdot 4^2 = 36^2 \cdot (3^2 + 4^2) = 36^2 \cdot 5^2 = (36 \cdot 5)^2 = 180^2$. We can rewrite the right side of the equation to $45^2 \cdot N^2 = (45 \cdot N)^2$. So $180 = 45 \cdot N$, so $N = \boxed{4}$.

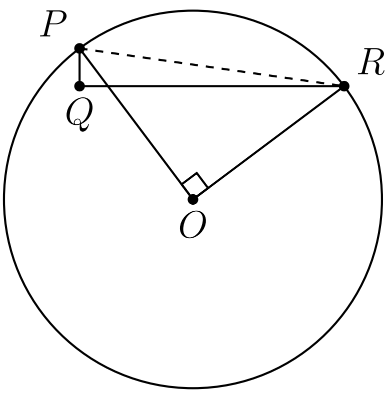
7. [5 points] We know that $YX = 4$. But as X is the midpoint of AB which is the diameter of the semicircle, we know that X is indeed the centre of the semicircle. Thus $XA = XY = XB = 4$. Now $XY = XA = 4 = AP$, so XAP is an isosceles triangle. Drawing a perpendicular from A to PX and calling this point D, we know that AP is perpendicular to PX, D bisects PX and thus $DX = 1$. So by Pythagoras $AD = \sqrt{4^2 - 1^2} = \sqrt{15}$ and $AY = \sqrt{AD^2 + YD^2} = \sqrt{15 + 9} = \sqrt{24} = \boxed{2\sqrt{6}}$.



8. [5 points] The total number of ways the four playing cards could be arranged is $4! = 24$. Now there are only 2 ways where the first card's word contains the letter N, the second word contains O, the third word contains A, and the fourth word contains H. Namely $\boxed{\text{NO}}\boxed{\text{OH}}\boxed{\text{AN}}\boxed{\text{HA}}$ and $\boxed{\text{AN}}\boxed{\text{NO}}\boxed{\text{HA}}\boxed{\text{OH}}$. Thus the probability is $\frac{2}{24} = \boxed{\frac{1}{12}}$.

9. [5 points] The n^{th} term of the arithmetic sequence is given by the formula $a_n = 1 + 7 \cdot (n - 1)$. Suppose we group the members of the arithmetic sequence as follows: $\{1\}$, $\{8\}$, $\{15\}$, $\{22\}$, $\{29, 1996\}$, $\{36, 1989\}$, $\ldots$, $\{1 + 7 \cdot i, 2024 - 7 \cdot i\}$, $\ldots$, $\{1009, 1016\}$. Note we constructed 145 groups. Now if there are two numbers in the same group within the n numbers we picked from S, there will necessarily be a pair of numbers that add up to 2025. So (by the pigeonhole principle) the smallest n we can pick to guarantee there to be a pair of numbers that add to 2025 is $\boxed{146}$.

10. [5 points] $PR = \sqrt{50}$. If we denote the centre of the circle to be O, we know that $OP = OR = 5$, so by the converse of the Pythagorean theorem, $\angle POR = 90°$. After some gruelling calculations, we get that the area of the two regions is $\frac{25}{4}\pi - 9$ and $\frac{75}{4}\pi + 9$; the smaller area is $\boxed{\frac{25}{4}\pi - 9}$.



11. [5 points] Cycling to class takes 30 minutes. Jogging from his home to the bus stop takes 10 minutes. Taking the bus from the bus stop to his maths class takes 18 minutes. So for the bus to get him to class faster than cycling, Arav would need the bus to arrive within 2 minutes of him getting to the bus stop. As the bus arrives every 5 minutes, and Arav leaves his house at a random time, the probability is thus $\boxed{\frac{2}{5}}$.

12. [5 points] There are 2 cases where Jimbo takes exactly two minutes to draw the triangle: Either he starts with (4,5) and ends with (1,2,3,4,5,6) or he starts with (1,2,3) and ends with (4,5,6). The probability is thus $\frac{(2 \cdot 6 + 3 \cdot 3)}{36} = \boxed{\frac{7}{12}}$.

13. [6 points] For $\frac{1}{n}$ to have a cycle of 3 digits, we know n is a factor of 999 as $\frac{1}{999} = 0.001001001\ldots$. The factors of 999 are 1, 3, 9, 27, 37, 111, 333, and 999. A quick check will tell us that only $n = 27, 37, 111, 333$, and 999 satisfy the property that $\frac{1}{n}$ has its shortest cycle being 3 digits long. Thus the answer is $\boxed{5}$.

14. [6 points] Note $\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2n-1) = n^2$. So, the numerator is equal to $1^2 \times 2^2 \times \cdots \times 13^2$. So after some cancellations, what's left on the denominator will be the primes greater than 13. So the answer is $17 \times 19 = \boxed{323}$.

15. [6 points] As given in the question we know that $a_{2n-1} = 2n - 1$ and

$$a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n}$$

and in particular, we know

$$\begin{aligned} a_1 + \cdots + a_{50} &= a_{51} + \cdots + a_{100} \\ a_1 + \cdots + a_{49} &= a_{50} + \cdots + a_{98} \end{aligned}$$

Now,

$$\begin{aligned} a_{100} &= a_1 + \cdots + a_{50} - (a_{51} + \cdots + a_{99}) \\ &= a_1 + \cdots + a_{49} - (a_{50} + \cdots + a_{98}) + 2a_{50} - a_{99} \\ &= 2a_{50} - a_{99} \end{aligned}$$

Similarly, we can obtain $a_{50} = 2a_{25} - 49 = 1$. Thus, $a_{100} = \boxed{-97}$.

16. [6 points] Count. Note the number of different ways Angelina can choose her outfits from Thursday to Saturday is the same number of ways she can choose her outfits from Monday to Wednesday (as each date's outfit only depends on the last date and we are working over the same time frame of 3 days). Suppose we labeled the outfits A, B, C, and D. There are 3 choices of outfits she can wear on Sunday. Without loss of generality, let her wear outfit A on Sunday. Then on Monday she can wear any of B, C, and D. If she wears outfit A on Tuesday, she can wear any of B, C, D on Wednesday for her date. But if she wears another outfit on Tuesday, then she only has 2 options to choose from on Wednesday (as she also cannot wear outfit A on Wednesday). Hence, the number of different ways she can choose her outfit from Monday to Wednesday is $3 \cdot (1 \cdot 3 + 2 \cdot 2) = 21$. Thus, the number of different ways she can choose her outfit from Sunday to Saturday is $3 \cdot 21 \cdot 21 = \boxed{1323}$.

17. [6 points] First note that $a^4 - b^4 = (a^2+b^2)(a+b)(a-b)$, so that 1, $a-b$, $a+b$, a^2+b^2, $(a+b)(a-b)$, $(a^2+b^2)(a-b)$, $(a^2+b^2)(a+b)$, $(a^2+b^2)(a+b)(a-b) = a^4-b^4$ are 8 numbers that divide a^4-b^4, which has 6 factors so that these numbers can't all be distinct. As $0 < a-b < a+b < a^2+b^2$, the only way for this to happen is for $a-b=1$. This leaves us with 8 possible solutions. Checking each of these solutions, we find that (a,b) is either $(4,3)$ or $(5,4)$ and so our answer is $175+369 = \boxed{544}$.

18. [6 points] Suppose we denote the bottom right vertex of the right triangle as A and the bottom left vertex as B. Let O_1 be the centre of the largest circle with radius 1, and O_2 be the centre of the second largest circle with radius r. Let D and E be where the circle centred at O_1 touches AB and where the circle centred at O_2 touches AB respectively. Note O_1, O_2, and A are collinear. $AD = \frac{17}{5} - 1 = \frac{12}{5}$ so by Pythagoras $O_1A = \frac{13}{5}$. Hence, $O_2A = \frac{13}{5} - 1 - r = \frac{8}{5} - r$. But as O_1AD and O_2AE are similar triangles (by AAA) we must have that $\dfrac{O_1A}{O_1D} = \dfrac{O_2A}{O_2E}$, so plugging in the numbers we have $\frac{13}{5} = \frac{\frac{8}{5}-r}{r}$. Solving for r yields $r = \frac{4}{9}$. Now this is the ratio in radii between any two consecutive circles, and thus the total area of all the circles is given by $\pi + \frac{4}{9}\pi + \left(\frac{4}{9}\right)^2\pi + \ldots$. As this is the sum of a geometric series with ratio $\frac{16}{81}$ we get that the sum of the areas of all the circles is $\dfrac{\pi}{1-\frac{16}{81}} = \boxed{\dfrac{81}{65}\pi}$.

19. [7 points] By log rules

$$\frac{\log_{10}(s_1) + \log_{10}(s_2) + \cdots + \log_{10}(s_n)}{\log_{10}(2025)} = \frac{\log_{10}(s_1 s_2 \ldots s_n)}{\log_{10}(2025)}$$

Now $2025 = 3^4 \cdot 5^2$ so 2025 has $(4+1)(2+1) = 15$ factors. Each factor 'pairs' up with another factor to give a product of 2025 except for 45 as $2025 = 45^2$. So $s_1 s_2 \ldots s_n = 2025^7 \cdot 45 = 45^{15}$. So

$$\begin{aligned}\frac{\log_{10}(s_1 s_2 \ldots s_n)}{\log_{10}(2025)} &= \frac{\log_{10}(45^{15})}{\log_{10}(45^2)} \\ &= \frac{15\log_{10}(45)}{2\log_{10}(45)} \\ &= \boxed{\frac{15}{2}}.\end{aligned}$$

20. [7 points] Aryan and Bao must walk a total of 3 km, which would take them $3000/2 = 1500$ seconds, equal to $1500/60 = 25$ minutes. So, the gates on both sides of the bridge must open fast enough so that they have more than 25 minutes between the both of them to walk across. Between the two of them, they have 60 minutes to spend, so that they can spend less than 35 minutes waiting for the gates to open. Let $W(x)$ be the number of minutes that the person on the west-side has to wait, and similarly let $E(x)$ be the number of minutes that the person on the east-side has to wait. The answer is the number of non-negative integers $x < 60$ such that $W(x)+E(x) < 35$. Using the fact that $W(x)$ is the remainder when x is divided by 20 and $E(x)$ is the remainder when x is divided by 30, we find that $W(x) + E(x) < 35$ exactly when x is one of $0, \dots, 17,\ 20, \dots, 27,\ 30, \dots, 52$. Our answer is therefore $\boxed{49}$.

21. [7 points] Consider the average price of delivery for n pizzas which is given by

$$f(n) = \frac{n + 1 - \lfloor \sqrt{n} \rfloor}{n} = 1 + \frac{1}{n} - \frac{\lfloor \sqrt{n} \rfloor}{n}$$

Setting $n = k^2 + m$ where $0 \leq m < 2k + 1$ we get

$$f(k^2 + m) = 1 + \frac{1}{k^2 + m} - \frac{k}{k^2 + m} = 1 - \frac{k-1}{k^2 + m} \geq 1 - \frac{k-1}{k^2}$$

Consider

$$\frac{k-1}{k^2} \geq \frac{k}{(k+1)^2}$$

This is true for $k \geq 2$. So we know for all $k \geq 2$ that $\frac{1}{4} \geq \frac{(k-1)}{k^2}$ with equality if and only if $k = 2$. Checking $k = 1$, we can see that $\frac{1-1}{1} \leq \frac{1}{4}$, so indeed over the positive integers, the maximum of $\frac{k}{(k+1)^2}$ is $\frac{1}{4}$.

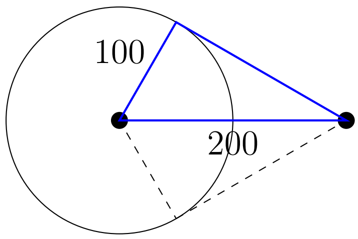
Thus, $1 - \frac{k-1}{k^2}$ achieves its minimum when $k = 2$, ie. when $n = 4$. So, the minimum amount that Julian spends on 100 pizzas will be when he buys 4 pizzas at a time, giving him a grand total of $100 \cdot 10 + \frac{3}{4} \cdot 100 = \boxed{1075}$.

22. [8 points] Swap the order of summation. Consider a fixed $n \geq 2$. Each term $\sigma(s)$ of the sum of Bernie's functions will contain $\frac{1}{n^s}$, and adding these up we find that these terms contribute

$$\sum_{s=2}^{\infty} \frac{1}{n^s} = \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots$$
$$= \frac{1/n^2}{1 - 1/n} \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

We can now look at the 'partial sums' of Bernie's functions, where we add up the contributions from each n. Starting with $n = 2$, we get $1 - \frac{1}{2}$. Adding in the contribution from $n = 3$ yields $1 - \frac{1}{3}$. Continuing in this way, we see that for any $k > 2$, the sum from $n = 2$ up to $n = k$ is $1 - \frac{1}{k}$. As k grows larger, $\frac{1}{k}$ approaches 0 and hence the sum of Bernie's functions approaches $\boxed{1}$.

23. [8 points] Because Sam and Owen walk at constant rates, then from Sam's frame of reference, Owen is moving at a constant rate in a random direction. The rate at which Owen moves relative to Sam is unimportant, as we only need to know in what direction Owen needs to move at in order to eventually get within a distance of 100 m of Sam. By drawing a circle of radius 100 m around Sam, we find that the lines intersecting Owen's position that are tangent to the circle form a 60° angle, since the blue right triangle shown below has the side length ratio of a half equilateral triangle.



This means that in Sam's reference frame, as long as Owen moves in a direction within a 60° window, then he and Sam will see each other. This means the probability is $\dfrac{60}{360} = \boxed{\dfrac{1}{6}}$.

24. [8 points] Let us extend the semicircle into a full circle and let us call this circle ω. Suppose YX intersects ω at another point Q. Then by power of a point $YP \cdot PQ = AP \cdot PZ$. Hence, $PQ = 6$ and thus $XQ = 4$. But $XQ = 4 = XY$ and X is not the centre of the circle (as $AX \neq BX$), Thus, YQ must be perpendicular to AB. Applying Pythagoras, we get $AX = \sqrt{4^2 - 2^2} = \sqrt{12}$ and by Pythagoras again, $AY = \sqrt{\sqrt{12}^2 + 4^2} = \sqrt{28} = \boxed{2\sqrt{7}}$.

25. [16 points] 192.

26. [16 points] 685126.

27. [16 points] Hopefully you remembered that each team's score on the OMTC 2025 is out of 700 points, and used the fact that there were 20 teams competing in-person (or 40 teams online). By thoroughly vibechecking the difficulty of the competition, you would find that the total number of points scored before the final question was $\boxed{3402.35}$ (in-person) or $\boxed{8508.6}$ (online).