Oxford Mathematics Team Challenge
Lock-in Round Solutions (with Questions)

Saturday, 8th March 2025

At the start of the next page are solutions to each question of the Lock-in Round.

Errata

Unfortunately this year's Lock-in Round had minor errors in one of the questions. We apologise for any confusion caused. The list of errata follows:

- **4. (e)** had typos in the programs 4. and 5.

All errors above have been amended by modifying the questions and solutions appropriately; the tests we have released online are updated accordingly.

1 Circle packing

In this question, we will look at the circle-packing problem. In the circle-packing problem, we are given a shape and have to "pack" a certain number of congruent circles inside it: we must place the circles without overlap in the interior of the shape. The goal of the problem is to maximise the radius of the circles.

(a) Figure 1 shows a packing of six circles in a larger circle of radius 3.

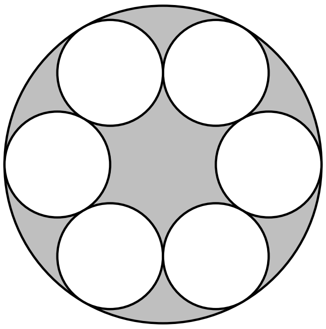
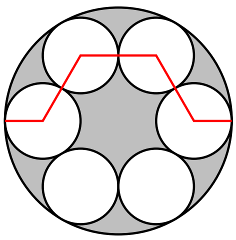


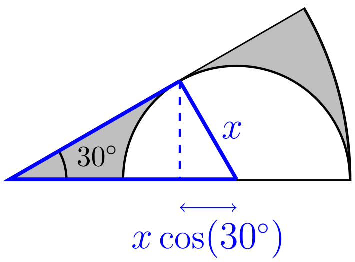
Figure 1

Carefully find the radius of the smaller circles. [3]

Sol. Let x be the radius we wish to find. We can solve for the diameter of the larger circle by considering the red lines:



The horizontal segments add up to $4x$, so we need to calculate the horizontal component of the diagonal segments. Zooming in to a subsection:

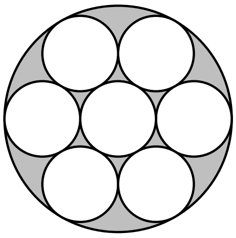


So the horizontal components of the diagonals sum to $4x\cos(30°) = 2x$. Since we know the diameter of the larger circle is 6, and we've just shown that this diameter is also $4x + 2x = 6x$, it follows that $x = 1$.

The packing of the six circles in Figure 1 is *optimal*, meaning that there is no way to fit six congruent circles of a larger radius into a circle of radius 3.

(b) Construct an optimal packing of seven circles in a circle of radius 3. [1]

Sol. Because the radius is 1, we can fit a seventh circle in the 6-packing we were given:

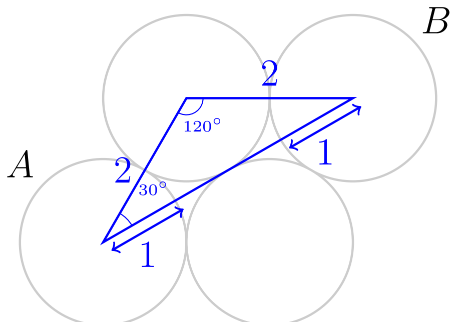


Given two packings of n circles inside a shape, we say that these packings are *equivalent* if we can essentially transform one packing into the other through a series of transformations which slide the circles or rotate the whole packing.

(c) Are all optimal packings of six circles in a circle equivalent? Explain your answer. [3]

Sol. There are inequivalent optimal packings of six circles. From (b), we can consider the following two packings:

These two packings are equivalent if we can slide C from its place in the left figure to its place in the right figure. We can do so only if the diameter of C (which is 2) is less than the shortest distance between A and B.



We can find the shortest distance between the circles A and B (see the figure on the previous page) by the sine rule to be

$$\frac{2\sin(120°)}{\sin(30°)} - 2 = 2\sqrt{3} - 2$$

Since $2\sqrt{3} - 2 < 2$, we can't slide C in between them. Therefore, we have two inequivalent optimal packings.

(d) An inconspicuous aside: Figure 2 depicts a triangle with side lengths 13, 13 and 10.

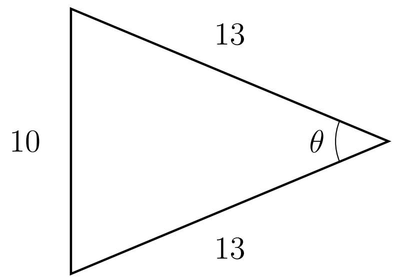
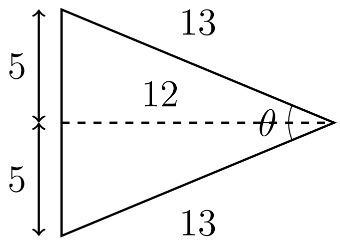


Figure 2

Show that θ, the angle between the sides of length 13, is approximately $45°$. Justify your working clearly. [3]

Sol. You can use the cosine rule, but using the sine area rule here is quite nice. Splitting the triangle in two:



We get the 5-12-13 Pythagorean triple, so the area is $2 \times (\frac{1}{2} \times 5 \times 12) = 60$. With the sine area rule, the area is $\frac{1}{2} \times 13^2 \sin\theta$, so equating the two:

$$\frac{1}{2} \times 13^2 \sin\theta = 60$$
$$169 \sin\theta = 120$$
$$\therefore \sin\theta = 120/169$$

By calculation, $120/169 = 0.710...$ which we can compare with $\sqrt{2}/2$: $\sqrt{2} = 1.414...$ so

$$\sqrt{2}/2 = 0.707... \approx 0.710... = 120/169$$

thus $\theta \approx 45°$.

(e) Figure 3 shows an optimal packing of nine circles in a circle of radius 3.

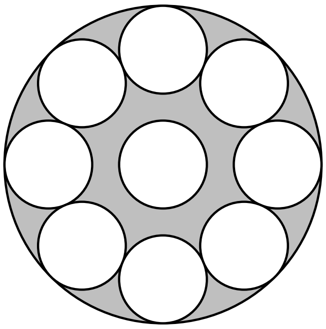
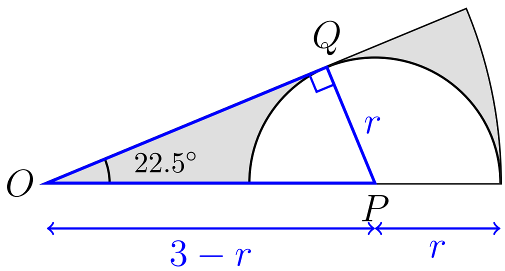


Figure 3

Find the radius r of the smaller circles in terms of $\sin(22.5°)$. [5]

Sol. Take a sixteenth of the diagram and zoom in. For clarity in the diagram we remove the centre circle. Let r be the radii of the smaller circles.



Note that $\angle OQP = 90°$ because the line OQ is tangent to the radius PQ, so we can do some trigonometry on the triangle OPQ to make an equation relating r and $\sin(22.5°)$:

$$
\begin{aligned}
\sin(22.5^\circ) &= \frac{r}{3-r} \\
(3-r)\sin(22.5^\circ) &= r \\
3\sin(22.5^\circ) &= r(1+\sin(22.5^\circ)) \\
\therefore r &= \frac{3\sin(22.5^\circ)}{1+\sin(22.5)}
\end{aligned}
$$

(f) Now suppose the central circle in Figure 3 was enlarged so it is just touching the surrounding circles, which is depicted in Figure 4.

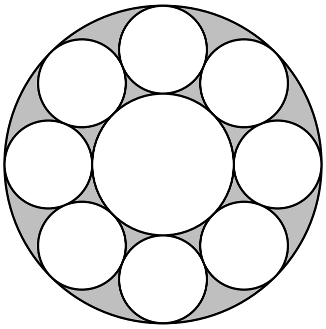


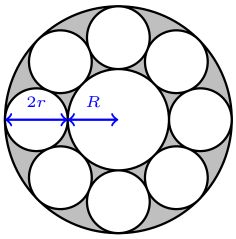
Figure 4

Let the central circle have radius R. Estimate the values of r and R. *[Hint: use (d) and (e).]* [5]

Sol. Using the figure from (d), we have an estimate for $\sin(22.5^\circ)$. The dashed line bisects θ, which we estimated to be 45°; so $\sin(22.5^\circ) \approx 5/13$. We substitute this into r:

$$
r \approx \frac{3 \times \frac{5}{13}}{1 + \frac{5}{13}} = \frac{3 \times 5}{13 + 5} = \frac{15}{18} = \frac{5}{6}
$$

Lastly, R equals $3 - 2r$ (see figure below), so $R \approx 4/3$.



(g) Hence, or otherwise, verify that the ratio of the areas between the central circle and the smaller circles equals 2.6 when rounded to the nearest tenth. [2]

Sol. The ratio of the areas is R^2/r^2, so

$$\frac{R^2}{r^2} = \frac{\frac{16}{9}}{\frac{25}{36}} = \frac{64}{25} = 2.56 \approx 2.6.$$

(h) Andrew wants to make a wedding cake by following his cake-making procedure:

Step A: Bake a cylindrical base with both radius and height 3 feet.

Step B: On the top of the cake, draw circular outlines as in Figure 4.

Step C: For each circular outline produced from the last step B, repeat this process, setting the radius and height in step A to the outline's radius.

In terms of R, what will the height of Andrew's cake be? [3]

Sol. The highest point in the cake will be from the central spire of the cake, so we should find that height. It'll sum to an infinite geometric series with first term 3 and ratio $R/3$, so by the infinite geometric series formula, the height is

$$h = \frac{3}{1 - \frac{R}{3}} = \frac{9}{3 - R}.$$

(i) Find an expression for the volume of Andrew's cake in terms of R and r. [5]

Sol. We can use the fact that each sub-cake is similar to the entire cake. On the first step, the central spire is similar to all of Andrew's cake but has a length scale-factor of $R/3$ (as in part (g)), so its volume will be scaled by a factor $(R/3)^3$. The remaining eight spires have a length scale-factor of $r/3$, so their volume will be scaled by $(r/3)^3$. The base cake has volume 27π, so:

$$V = 27\pi + V\left(\frac{R}{3}\right)^3 + 8V\left(\frac{r}{3}\right)^3$$

By collecting the V terms and factorising, we get that

$$V = \frac{729\pi}{27 - R^3 - 8r^3}.$$

2 Integer partitions

In this question, we will explore *partition theory*. A *partition* of a positive integer n is a way of writing n as the sum of positive integers (called the *parts*), irrespective of the order of the sum. For example, $5+2+1$ is a partition of 8; this is the same partition as $2+1+5$, however a different partition of 8 is e.g. $3+3+2$. A sum with only one part counts as a partition, e.g. 5 is a partition of 5.

(a) Write down all the partitions of 4. [1]

> **Sol.** The partitions of 4 are:
>
> $$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1.$$

(b) (i) A partition is called *distinct* if no part is repeated in the sum. For example, $5+2+1$ is a distinct partition of 8, whereas $3+3+2$ is not.

Write down all distinct partitions of 7. [2]

> **Sol.** The distinct partitions of 7 are:
>
> $$7, \qquad 6+1, \qquad 5+2, \qquad 4+3, \qquad 4+2+1.$$

(ii) A partition is called *odd* if it only contains odd parts. For example, $5+3$ is an odd partition of 8, whereas $5+2+1$ is not.

Write down all odd partitions of 7. [2]

> **Sol.** The odd partitions of 7 are:
>
> $$7, \quad 5+1+1, \quad 3+3+1, \quad 3+1+1+1+1, \quad 1+1+1+1+1+1+1.$$

(c) For a positive integer n, let $p(n)$ be the number of partitions of n. By convention we also say $p(0)=1$.

(i) Calculate $p(5)$. [1]

Sol. $p(5) = 7$ – the best way to see this is just to exhaust all partitions of 5:

$$5, \qquad 4+1, \qquad 3+2, \qquad 3+1+1,$$
$$2+2+1, \qquad 2+1+1+1, \qquad 1+1+1+1+1.$$

(ii) Explain why $p(n+1) > p(n)$ for all $n \geq 1$. [2]

Sol. For each partition of n, adding one (as a separate part) results in a unique partition of $n+1$, so at least $p(n+1) \geq p(n)$. These partitions don't include the partition of one part, $n+1$, so $p(n+1) > p(n)$.

(iii) We also write $p_d(n)$ as the number of distinct partitions of n, and $p_o(n)$ as the number of odd partitions of n. Similarly, $p_d(0) = p_o(0) = 1$.

Calculate $p_d(5)$ and $p_o(5)$. [2]

Sol. We have $p_o(5) = p_d(5) = 3$. In fact, you only needed to check one of these, because looking ahead to (e)(ii) we later show that $p_o(n) = p_d(n)$ for all n!

In any case, the odd partitions of 5 are

$$5, \qquad 3+1+1, \qquad 1+1+1+1+1.$$

and the distinct partitions of 5 are

$$5, \qquad 4+1, \qquad 3+2.$$

We can draw out a partition using a Ferrer diagram, which represents the partition as a collection of dots, descending in size of the rows. Figure 5 shows two examples:

$6+3+3+2+1$ $\quad$ $5+4+3+1+1+1$

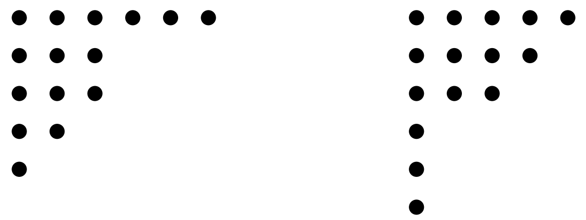
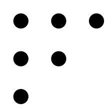


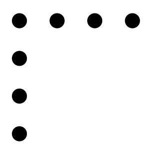
Figure 5

For any partition, we can create its *conjugate* by reflecting the Ferrer diagram over the diagonal. For example, the partitions in Figure 5 are conjugates of each other. We also say that a partition is *self-conjugate* if its conjugate is itself.

(d) (i) Give an example of a self-conjugate partition of 6. [1]

> **Sol.** A partition being self-conjugate is equivalent to a partition's Ferrer diagram having reflective symmetry along the diagonal. The only self-conjugate partition of 6 is $3+2+1$:
>
> 

(ii) Give an example of a self-conjugate partition of 7. [1]

> **Sol.** The only self-conjugate partition of 7 is $4+1+1+1$:
>
> 

(iii) Explain why the number of partitions with six parts is the same as the number of partitions with the largest part equal to 6. [4]

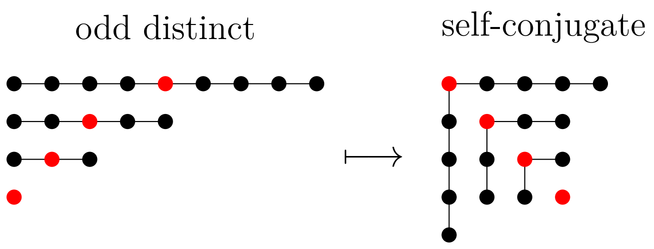
> **Sol.** The conjugate operation creates a one-to-one correspondence between partitions (potentially mapping a partition to itself, if the partition is self-conjugate). We can also think of the conjugate as reading the partition column-wise, as opposed to row-wise (the normal way).

If the largest part of a partition is 6, the Ferrer diagram has six columns, so its conjugate will have six parts. Conversely, if a partition has six parts, the Ferrer diagram has six rows; as the Ferrer diagram descends in the size of rows, its tallest column must be 6, i.e. the conjugate has largest part equal to 6.

This establishes a one-to-one correspondence between partitions with largest part equal to 6, and partitions with six parts, hence they are equal in number for a partition of any n.

(iv) Explain why the number of partitions of n into distinct odd parts is the same number of partitions of n into self-conjugate parts. [4]

Sol. Start with an odd distinct partition. Take the middle point in each row and use it as a "hinge" for a new Ferrer diagram:



The odd distinct partition forms a self-conjugate diagram as each hinge reflects along the diagonal. This is also a *valid Ferrer diagram* as each part in the original partition is distinct (each hinge needs to be at least 2 greater than the last, which is guaranteed by the partition being distinct)!

We can go backwards, too, as each self-conjugate partition gives us the hinges along the diagonal. It's odd because there's an equal number of points to the right of and below each hinge; it's Ferrer, and moreover distinct, because the groupings form a strictly decreasing sequence. This establishes a one-to-one correspondence between odd distinct partitions and self-conjugate partitions, hence they are equal in number for a partition of any n.

(e) Let's return to $p_o(n)$ and $p_d(n)$.

(i) Consider the infinite products

$$A = (1+x)(1+x^2)(1+x^3)\cdots$$
$$B = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

Explain and justify how the coefficients of x^n in A and B relate to $p_d(n)$ and $p_o(n)$. [5]

Sol. Recall that we defined

$$A = (1+x)(1+x^2)(1+x^3)\cdots$$
$$B = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

We can associate the coefficient of x^n in A to $p_d(n)$, and the coefficient of x^n in B to $p_o(n)$. Why? Consider $p_d(3)$, for example. The only distinct partitions for 3 are 3 and $2+1$. Now consider the coefficient of x^n in A. We can find it by thinking what we need to multiply together to make a term of x^3; e.g., we can choose

$$(\mathbf{1}+x)(\mathbf{1}+x^2)(1+\boldsymbol{x^3})(\mathbf{1}+x^4)\cdots$$

or

$$(1+\boldsymbol{x})(1+\boldsymbol{x^2})(\mathbf{1}+x^3)(\mathbf{1}+x^4)\cdots$$

These correspond to the distinct partitions 3 and $1+2$, respectively. For example, choosing to multiply by the x^2 is the same as including 2 as a part in your partition; choosing to multiply by 1 instead of x^2 is the same as not including 2 in your partition.

This generalises to any x^n in A: its coefficient tells us the number of distinct partitions.

The same goes for B with odd partitions. The first bracket tells us the number of 1's we add to the sum; the second bracket tells us the number of 3's we add to the sum; and so on.

(ii) Show that $p_d(n) = p_o(n)$. *[Hint: consider* $\dfrac{1-x^{2k}}{1-x^k}$ *for each positive integer* k*.]*

[5]

Sol. Time to crunch some algebra! The hint tells us to consider $\dfrac{1-x^{2k}}{1-x^k}$. On the one hand,

$$\frac{1-x^{2k}}{1-x^k} = \frac{(1-x^k)(1+x^k)}{1-x^k} = (1+x^k)$$

so we can think of A as the infinite product

$$\begin{aligned} A &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\ &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdot\frac{1-x^{10}}{1-x^5}\cdots \end{aligned}$$

We can see that all of the numerators $(1-x^{2k})$ will eventually cancel with a denominator later on:

$$= \frac{\cancel{1-x^2}}{1-x}\cdot\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\cdot\frac{\cancel{1-x^6}}{1-x^3}\cdot\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\cdot\frac{\cancel{1-x^{10}}}{1-x^5}\cdots$$

This leaves us with A equal to

$$A = \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots$$

These are each infinite geometric sums, with first term 1 and ratio x, x^3, x^5, and so on respectively. That is,

$$\begin{aligned} &= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots \\ &= B. \end{aligned}$$

Remark. This was a lovely proof which Leonhard Euler came up with in 1748 – that the number of distinct partitions of n equals the number of odd partitions of n, and that this can be proved using these types of infinite products which we call *generating functions*.

3 Random tic-tac-toe

Two robots, Xeep and Obot, play a game of tic-tac-toe (a.k.a. noughts and crosses) on a 3×3 grid. Xeep, who goes first, marks its squares with Xs, and Obot marks with Os. They take turns choosing a random unfilled square to mark. Each robot is equally likely to choose any one of the unfilled squares on its turn. A robot wins by being the first to achieve a 3-in-a-row of their own symbols, either horizontally, vertically, or diagonally. If the grid gets filled completely with no such 3-in-a-row, then the game ends in a draw.

We give the robots additional instructions so that even if a robot has won before the grid is completely filled, the robots will continue to make random moves until the grid is completely filled. We shall call the final completely-filled grid of this game the *end grid*. Note that every possible end grid has 5 Xs and 4 Os.

The following questions lead to finding the probability of a draw.

(a) Briefly describe how you can determine whether or not a game of tic-tac-toe ended in a draw if you are provided the end grid of the game. [2]

Sol. A game of tic-tac-toe ended in a draw if and only if the end grid does not contain any 3-in-a-rows of all Xs or all Os. If there was a 3-in-a-row, then at some point during the game, the robot would complete that 3-in-a-row, so that game ends with a robot winning.

(b) Given an end grid of a game that didn't end in a draw, is it always possible to determine who the winner was? Briefly explain. [3]

Sol. Given an end grid of a game that didn't end in a draw, we cannot always determine who the winner was. If we get an end grid with a 3-in-a-row from both X and O, such as the grid shown below, then either Xeep or Obot could've won this game, depending on whose 3-in-a-row was formed first in the sequence.

X	O	X
X	O	X
X	O	O

(c) There are exactly 126 possible end grids. What is the probability that a game of random tic-tac-toe ends in a draw? Express your answer as a simplified fraction. *[Hint: find the number of end grids that could result from a drawn game.]* [7]

Sol. Since all legal moves are made with equal probability, the resulting end grid of a game of random tic-tac-toe is equally likely to be any one of the 126 possible end grids. From (a), the end grid indicates a drawn game if there is no 3-in-a-row. So if we count the number of drawn end grids, then divide that by 126, that result is the proportion of drawn end grids, which is the probability that a game of random tic-tac-toe ended in a draw.

The big task is counting the number of drawn end grids, i.e. those without a 3-in-a-row. There are various systems you may use to keep track of your grids, to make sure you don't miss any cases or count the same grid twice. The system we show here is just one way to approach this.

We approach this by divide and conquer. We'll separate the end grids into smaller buckets, and then calculate the number of drawn end grids within each smaller bucket. In any end grid, there are either 0, 1, 2, 3, or 4 Os at the corners, so one way you may divide and conquer this problem is to consider cases of the number of Os at the corners.

Four Os at the corners. Recall that all end grids have 4 Os and 5 Xs. There is only 1 end grid with 4 Os at the corners, and it has a 3-in-a-row of Xs, as shown below. So there are **0** drawn end grids in this case.

O	X	O
X	X	X
O	X	O

Three Os at the corners. We draw in the corners first, shown below on the left. To avoid making a 3-in-a-row, we're forced to draw 3 more Xs between the Os as shown on the right.

O		O
O		X

O	X	O
X	X	
O		X

But now we're stuck, because we need to place one more X, which will form a 3-in-a-row. So there are in fact **0** drawn end grids in this case.

Two Os at the corners. We can't have the Os at opposite corners because then no matter our choice for the centre square, we will form a 3-in-a-row. Therefore, we'll consider arrangements with the Os at neighbouring corners.

O		X
X		O

O		O
X		X

In the grid on the right, we force the top-middle to be X and the bottom-middle to be O. We have one more O to place in the grid, and it turns out that any of the placements results in a drawn end grid!

O	X	O
X	O	X

⟶

O	X	O
O	X	X
X	O	X

O	X	O
X	O	X
X	O	X

O	X	O
X	X	O
X	O	X

Don't forget that we each rotate each of these grids to produce 4 distinct end grids. So in total, these are **12** drawn end grids in this case.

One O at the corners. Having 1 O and 3 Xs at the corners forces 3 more Os as shown below, then that forces the remaining squares to be Xs. This resulting grid has no 3-in-a-row, so it counts toward our total. Since this also can be rotated, there are **4** drawn end grids in this case.

O		X
X		X

O		X
	O	O
X	O	X

O	X	X
X	O	O
X	O	X

No Os at the corners. In this case, we place 4 Xs at the corners, but we need to place one more X, which cannot be done without forming a 3-in-a-row. So there are **0** drawn end grids in this case.

Any end grid falls into exactly one of the above cases, so we have accounted for all possible drawn end grids. The figure below shows the drawn grids we've found (note that any rotation of these grids is also a drawn grid).

O	X	O
O	X	X
X	O	X

O	X	O
X	O	X
X	O	X

O	X	O
X	X	O
X	O	X

O	X	X
X	O	O
X	O	X

Adding up the subtotals from each case, there are in total 16 drawn end grids. Therefore, the probability of random tic-tac-toe ending in a draw is 16/126, or 8/63.

The following questions lead to finding the probability that Xeep wins, or that Obot wins.

(d) How many end grids contain a 3-in-a-row of Os, but not a 3-in-a-row of Xs? [4]

> **Sol.** A 3-in-a-row of Os can run either vertically, horizontally, or diagonally. But notice how if we have our 3-in-a-row of Os run vertically, that leaves two more vertical rows and only one more O left to block them, so here we will also have a vertical 3-in-a-row of Xs. The same applies to a horizontal 3-in-a-row of Xs. In order to get an end grid with a 3-in-a-row of Os, but not a 3-in-a-row of Xs, we must have the Os run diagonally. There are 2 diagonals to choose from for our 3-in-a-row of Os, and then there are 6 spots remaining to place the final O. In total, there are $2 \times 6 = 12$ such end grids.

(e) How many end grids contain both a 3-in-a-row of Os and a 3-in-a-row of Xs? [4]

> **Sol.** Using the logic from (d), we now want our 3-in-a-row of Os to run either horizontally or vertically. There are 3 horizontal rows and 3 vertical rows to choose from, giving 6 options for where to place our 3-in-a-row of Os. Then there are 6 spots remaining from which to place our final O, and any choice also creates a 3-in-a-row of Xs. In total, there are $6 \times 6 = 36$ such end grids.

(f) Consider a variant of tic-tac-toe called *ric-rac-roe*, where parts of the grid are shaded grey as shown in Figure 6. In this variant, Xeep can only make moves in the white squares, and Obot can only make moves in the grey squares. Both robots are still equally likely to choose any one of its legal moves for each turn. What is the probability that Xeep wins in ric-rac-roe? Express your answer as a simplified fraction. *[Hint: consider the probability that Obot can achieve a 3-in-a-row in three moves.]* [6]

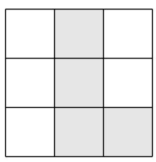


Figure 6

> **Sol.** There is a $1/4$ probability that Obot could achieve a 3-in-a-row in 3 moves, since after 3 moves, 1 grey square will be empty, and we need that to be the bottom-right square. There is a $1/10$ probability that Xeep gets a 3-in-a-row in 3 moves, since there are 10 total pairs of white squares, and we need the pair of white squares on the right to be empty after three moves. There is a $2/5$ probability that Xeep gets a 3-in-a-row

in 4 moves or fewer, since there would be 1 white square empty, and we need it to be one of the 2 white squares on the right.

If we let the game play out until the grid is filled, in the 1/4-probability case that Obot gets a 3-in-a-row in 3 moves, Xeep must hit its own 3-in-a-row in 3 moves in order to win. This case is a $\frac{1}{4} \times \frac{1}{10} = \frac{1}{40}$ probability win for Xeep. The other case is the 3/4-probability case is that Obot only gets a 3-in-a-row in exactly 4 moves. For Xeep to win it must hit a 3-in-a-row in 4 moves or fewer, so the probability for this case is $\frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$. If Xeep doesn't hit a 3-in-a-row in 4 moves or fewer then it must lose.

In total, the probability that Xeep wins ric-rac-roe is $\dfrac{1}{40} + \dfrac{3}{10} = \dfrac{13}{40}$.

(g) What is the probability that Xeep wins a game of random tic-tac-toe? What is the probability of Obot winning? Express your answers as simplified fractions. [4]

Sol. To find the probability Obot wins random tic-tac-toe, consider the end grids that could result in a win for Obot. In (d), we showed there are 12 grids with a 3-in-a-row for O and not for X. If a game has this end grid, we can be certain that Obot won this game. In (e), we showed there are 36 grids with a 3-in-a-row for both O and X, and in this case we cannot be certain of who won. In (f), which shows one such ambiguous case, there is a $1 - \frac{13}{40} = \frac{27}{40}$ probability that Obot wins if we assume that the final end grid has a 3-in-a-row for both O and X. So if we are given one of the 36 end grids with a 3-in-a-row for both players, we can say with $\frac{27}{40}$ certainty that Obot won that game.

Since no other end grids have a 3-in-a-row of Os, these are the only cases which Obot could win. The probability Obot wins in random tic-tac-toe is therefore

$$\frac{12}{126} + \frac{36}{126} \cdot \frac{27}{40} = \frac{121}{420}$$

The probabilities of Obot winning, of Xeep winning, and of a draw must add up to 1, since these are the only possible outcomes. Since from (c) the probability of a draw is $\frac{8}{63}$, the probability that Xeep wins is

$$1 - \frac{8}{63} - \frac{121}{420} = \frac{737}{1260}$$

4 FRACTRAN

FRACTRAN is a Turing-complete programming language invented by John Conway. Turing-complete means it can perform a lot of complex tasks, and by the end you'll be able to write some programs!

A FRACTRAN program consists of a finite list of positive fractions. The program acts as a function that takes in a positive integer input, and it will output a positive integer according to the following procedure:

1. Given an integer n as the current input, the program looks for the first fraction in the list whose denominator is a factor of n.
2. If the program finds this fraction, which we write as $\frac{a}{b}$, then it repeats step 1 with $n \cdot \frac{a}{b}$ as the new current input.
3. If there is no such fraction in the list, then the program outputs its current input and stops running.

As an example, consider the FRACTRAN program $\left[\frac{5}{3}, \frac{5}{2}\right]$. Let's run this with 6 as input. The first fraction whose denominator divides 6 is $\frac{5}{3}$, so we repeat step 1 with $6 \cdot \frac{5}{3} = 10$ as the new input. The first fraction whose denominator divides 10 is $\frac{5}{2}$, so we repeat step 1 with $10 \cdot \frac{5}{2} = 25$ as the new input. No fraction in the list has a denominator dividing 25, so we output 25 and stop. When given 6 as input, our program outputs 25.

(a) We can record each step of a FRACTRAN program in a *log*, which is a list of integers starting with the input and ending with the output, with all intermediate steps in between in order. When we input 6 into the program $\left[\frac{5}{3}, \frac{5}{2}\right]$, our log will be $(6, 10, 25)$.

Write the logs of $\left[\frac{5}{3}, \frac{5}{2}\right]$ when ran on each of the following inputs: 4, 18, and 24. [3]

Sol. We have the following logs:

$$\begin{aligned} 4 &\to (4, 10, 25) \\ 18 &\to (18, 30, 50, 125) \\ 24 &\to (24, 40, 100, 250, 625) \end{aligned}$$

(b) When $\left[\frac{5}{3}, \frac{5}{2}\right]$ runs on input $2^x \cdot 3^y$, where x and y are non-negative integers, what is the output? Express your answer in terms of x and y. [1]

Sol. Note that doing it only once results in $2^x 3^{y-1} 5$ if $y > 0$. If $y = 0$, then it results in $2^{x-1} 5$ if $x > 0$. If both x and y are 0, then no fraction in the program will be multiplied, and as such we would have reached the end of this program.

So, this leads us to see that it first converts all the 3's into 5's, and then converts all the 2's into 5's. This means we have a program that sends $2^x 3^y$ to 5^{x+y}.

(c) According to the rules of FRACTRAN stated above, we allow our programs to contain unsimplified fractions. In fact, unsimplified fractions can lead to different behaviour from their simplified equivalents. To demonstrate this, consider the two programs $\left[\frac{2}{3}\right]$ and $\left[\frac{4}{6}\right]$. Find a positive integer input such that these programs produce different outputs when run on that input. [2]

Sol. One example is simply plugging in 3, as under the first program it is sent to 2 and the second program keeps it at 3. More specifically, any positive integer with more factors of 3 than factors of 2 would work in this problem.

(d) It is possible for a FRACTRAN program to run forever when given a certain input, never stopping and therefore never outputting anything. Write a FRACTRAN
program and find a positive integer input such that this program will run forever when given this input. [2]

Sol. One example is the program $\left[\frac{1}{1}\right]$ and any positive integer n, as 1 divides all positive integers. If you want a program that runs infinitely on only certain inputs, note that $\left[\frac{k}{k}\right]$ runs infinitely on inputs of the form kn but halts on inputs of the form $kn + 1$ if $k > 1$.

(e) Match each FRACTRAN program listed on the left to exactly one description listed on the right. The program should achieve the specified output when given the specified input, where x and y are general non-negative integers. Express your answer as pairs of one number and one letter for each match.

1. $\left[\frac{1}{2}\right]$
2. $\left[\frac{9}{2}\right]$
3. $\left[\frac{3}{6}, \frac{9}{2}\right]$
4. $\left[\frac{3}{10}, \frac{5}{2}, \frac{1}{5}\right]$
5. $\left[\frac{15}{10}, \frac{3}{5}, \frac{10}{2}\right]$

A. input 2^x, output 3^{x+1}

B. input $2^x \cdot 3^y$, output 3^{2x+y}

C. input 2^{x+1}, output 9

D. input $2^x \cdot 5^y$, output 5^y

E. input 2^x, output $3^{x/2}$ if x is even and $3^{(x-1)/2}$ if x is odd [5]

Sol. We have the pairs (1, D), (2, B), (3, C), (4, E), and (5, A). Note that these proofs can be formalised with induction.

For the first program, note that it simply clears any powers of 2. The only program on the right that does this is D.

For the second program, note that for every 2 in the input, it is multiplied by 3^2, meaning that $2^x \to 3^{2x}$. So, the program that does this is B.

By now we only have A, C, and E remaining, so we only have to deal with problems with inputs that are powers of two.

So, for the third program, plugging in just 2 gives you 9. Plugging in 4 gives you the log $(4, 18, 9)$. More formally, plugging in 2^{x+1} gives you the log $(2^{x+1}, 3^2 2^x, 3^2 2^{x-1}, \ldots, 3^2)$. So, this is C.

For the fourth program, consider the log $(2^x, 2^{x-1}5, 2^{x-2}3, \ldots, 2^{x-2k}3^k)$. If x is even, then this will result in the output $3^{x/2}$. If x is odd, then we have the log end with $(2 \cdot 3^{(x-1)/2}, 3^{(x-1)/2}5, 3^{(x-1)/2})$, which tells us that this function is E.

Lastly, for the fifth program, we first compute it on 2 and get $(2, 10, 15, 9)$. Then consider the general log

$$(2^x, 2^x 5, 2^{x-1} \cdot 3 \cdot 5, 2^{x-2}3^2 5, \ldots, 2^{x-k}3^k 5, \ldots, 3^x 5, 3^{x+1})$$

So, the fifth program is A.

(f) Derrick claims that he created a FRACTRAN program such that for any positive integer input, it will output double the input. Briefly explain why Derrick's claim is impossible. [2]

Sol. Derrick's program would send x to $2x$, but it would also send $2x$ to $4x$. This means that some denominator in Derrick's program must divide $2x$, but this means that it can never terminate on $2x$, meaning that when x is put in as input, it will run forever. So, Derrick's program is impossible.

(g) Patrick claims that he created a FRACTRAN program that outputs 1 when given any positive integer input. Briefly explain why Patrick's claim is impossible. [3]

Sol. Every FRACTRAN program is finite, so consider the least common multiple of all of the denominators, call it L. Note if 1 is a denominator, then the program would never terminate, so we must have that $L + 1$ is coprime to all of the denominators. Also note that this implies that L is larger than 1. So, if I plugged in $L + 1$, it would not trigger on any denominator, and as such would not be sent to 1. So, Patrick's program is impossible.

(h) Write a FRACTRAN program such that for any positive integer x, an input of 2^x results in an output of 3 if x is even and an output of 5 if x is odd. [3]

Sol. We have the program

$$\left[\frac{3}{12}, \frac{5}{6}, \frac{3}{4}, \frac{5}{2}\right].$$

Note that when we plug in 2, it is sent to 5. Then, if we plug in 2^2, it is sent to 3. If we plug in 2^3, it is sent to 5. So, we can prove that this works by induction. Assume that for $2^{2k}3$ that it is sent to 3. Now, we consider:

$$(2^{2k+2}, 2^{2k}3, 2^{2(k-1)}3, \ldots, 3)$$

So, assume that for $2^{2k-1}3$ that it is sent to 6. We have the log:

$$(2^{2k+1}, 2^{2(k-1)+1}3, 2^{2(k-2)+1}3, \ldots, 2 \cdot 3, 5)$$

Note how we had to induct on something that was not just a power of 2.

(i) Write a FRACTRAN program such that for any non-negative integers x, y, and z, an input of $2^x \cdot 3^y \cdot 5^z$ results in an output of 7^m, where $m = \max\{x, y, z\}$, i.e. the maximum value of x, y, and z. [4]

Sol. There are a few possible solutions to this, but the simplest solution is of the form

$$\left[\frac{7}{30}, \frac{7}{6}, \frac{7}{10}, \frac{7}{15}, \frac{7}{2}, \frac{7}{3}, \frac{7}{5}\right].$$

Very simply put, the first fraction sends $2^x 3^y 5^z$ to $2^{x-m}3^{y-m}5^{z-m}7^m$ where $m = \min(x, y, z)$. Note then that we have cleared at least one factor. The rest of the program simply continues this. One can prove this then via casework and induction, using the equality

$$\max(x, y, z) = m + \max(x - m, y - m, z - m).$$

(j) Write a FRACTRAN program such that for any non-negative integers x and y, an input of $2^x 3^y$ results in an output of 5^{xy}. *[Hint: consider a program whose log begins with* $(2^x \cdot 3^y, 2^{x-1} \cdot 3^y \cdot 11, 2^{x-1} \cdot 3^{y-1} \cdot 5 \cdot 7 \cdot 11, \ldots, 2^{x-1} \cdot 5^y \cdot 7^y \cdot 11, 2^{x-1} \cdot 5^y \cdot 7^y, \ldots)$*.]* [5]

Sol. The program for this question is

$$\left[\frac{385}{33}, \frac{1}{11}, \frac{3}{7}, \frac{11}{2}, \frac{1}{3}\right].$$

Note $385 = 5 \cdot 7 \cdot 11$. Consider the log

$$\begin{aligned}(2^x 3^y, 2^{x-1}3^y 11, 2^{x-1}3^{y-1}5^1 7^1 11^2, \ldots\\ 2^{x-1}5^y 7^y 11^{y+1}, \ldots, 2^{x-1}5^y 7^y, \ldots\\ 2^{x-1}3^y 5^y, \ldots\\ 2^{x-k}3^y 5^{yk}, \ldots\\ 3^y 5^{xy}, \ldots\\ 5^{xy})\end{aligned}$$

As 5 does not actually appear in the denominator, one can induct on $2^{x-1}3^y$.