Oxford Mathematics Team Challenge
Lock-in Round Solutions

Saturday, 8$^{\text{th}}$ March 2025

At the start of the next page are solutions to each question of the Lock-in Round.

Errata

Unfortunately this year's Lock-in Round had minor errors in one of the questions. We apologise for any confusion caused. The list of errata follows:

- **4. (e)** had typos in the programs 4. and 5.

All errors above have been amended by modifying the questions and solutions appropriately; the tests we have released online are updated accordingly.

1 Circle packing

(a) [3 points] Let x be the radius we wish to find. We can solve for the diameter of the larger circle by considering the red lines:

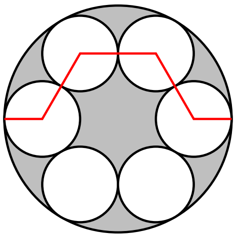


Figure 1

The horizontal segments add up to $4x$, so we need to calculate the horizontal component of the diagonal segments. Zooming in to a subsection:

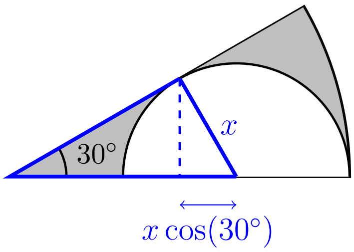


Figure 2

So the horizontal components of the diagonals sum to $4x\cos(30°) = 2x$. Since we know the diameter of the larger circle is 6, and we've just shown that this diameter is also $4x + 2x = 6x$, it follows that $x = 1$.

(b) [1 point] Because the radius is 1, we can fit a seventh circle in the 6-packing we were given:

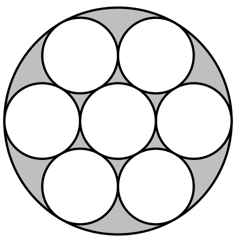


Figure 3

(c) [3 points] There are inequivalent optimal packings of six circles. From (b), we can consider the following two packings:

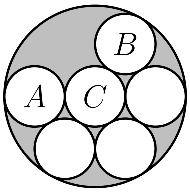


Figure 4

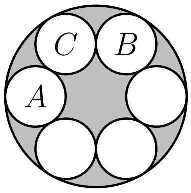


Figure 5

These two packings are equivalent if we can slide C from its place in Figure 4 to its place in Figure 5. We can do so only if the diameter of C (which is 2) is less than the shortest distance between A and B.

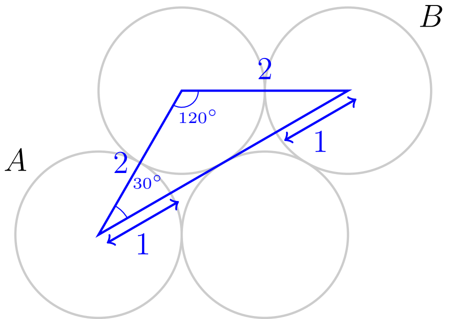


Figure 6

We can find the shortest distance between the circles A and B (see Figure 6) by the sine rule to be

$$\frac{2\sin(120°)}{\sin(30°)} - 2 = 2\sqrt{3} - 2$$

Since $2\sqrt{3} - 2 < 2$, we can't slide C in between them. Therefore, we have two inequivalent optimal packings.

(d) [3 points] You can use the cosine rule, but using the sine area rule here is quite nice. Splitting the triangle in two:

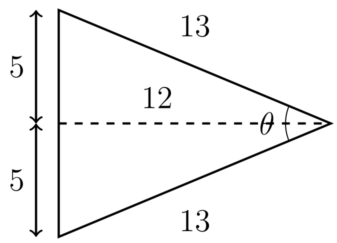


Figure 7

We get the 5-12-13 Pythagorean triple, so the area is $2 \times (\frac{1}{2} \times 5 \times 12) = 60$. With the sine area rule, the area is $\frac{1}{2} \times 13^2 \sin\theta$, so equating the two:

$$\begin{aligned} \tfrac{1}{2} \times 13^2 \sin\theta &= 60 \\ 169 \sin\theta &= 120 \\ \therefore \sin\theta &= 120/169 \end{aligned}$$

By calculation, $120/169 = 0.710...$ which we can compare with $\sqrt{2}/2$: $\sqrt{2} = 1.414...$ so

$$\sqrt{2}/2 = 0.707... \approx 0.710... = 120/169$$

thus $\theta \approx 45°$.

(e) [5 points] Take a sixteenth of the diagram and zoom in. For clarity in the diagram we remove the centre circle. Let r be the radii of the smaller circles.

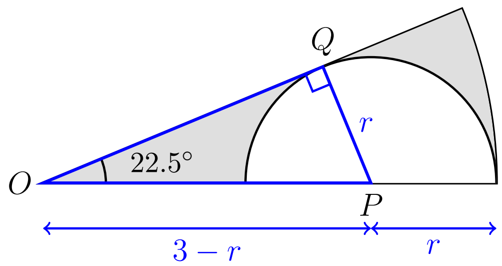


Figure 8

Note that $\angle OQP = 90°$ because the line OQ is tangent to the radius PQ, so we can do some trigonometry on the triangle OPQ to make an equation relating r and $\sin(22.5°)$:

$$\begin{aligned} \sin(22.5°) &= \frac{r}{3-r} \\ (3-r)\sin(22.5°) &= r \\ 3\sin(22.5°) &= r(1+\sin(22.5°)) \\ \therefore r &= \frac{3\sin(22.5°)}{1+\sin(22.5)} \end{aligned}$$

(f) [5 points] Using Figure 3 from (d), we have an estimate for $\sin(22.5°)$. The dashed line bisects θ, which we estimated to be $45°$; so $\sin(22.5°) \approx 5/13$. We substitute this into r:

$$r \approx \frac{3 \times \frac{5}{13}}{1 + \frac{5}{13}} = \frac{3 \times 5}{13 + 5} = \frac{15}{18} = \frac{5}{6}$$

Lastly, R equals $3 - 2r$ (see Figure 9), so $R \approx 4/3$.

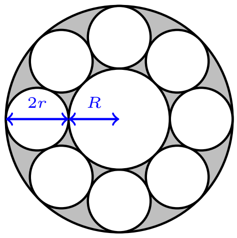


Figure 9

(g) [2 points] The ratio of the areas is R^2/r^2, so

$$\frac{R^2}{r^2} = \frac{\frac{16}{9}}{\frac{25}{36}} = \frac{64}{25} = 2.56 \approx 2.6.$$

(h) [3 points] The highest point in the cake will be from the central spire of the cake, so we should find that height. It'll sum to an infinite geometric series with first term 3 and ratio $R/3$, so by the infinite geometric series formula, the height is

$$h = \frac{3}{1 - \frac{R}{3}} = \frac{9}{3 - R}.$$

(i) [5 points] We can use the fact that each sub-cake is similar to the entire cake. On the first step, the central spire is similar to all of Andrew's cake but has a length scale-factor of $R/3$ (as in part (g)), so its volume will be scaled by a factor $(R/3)^3$. The remaining eight spires have a length scale-factor of $r/3$, so their volume will be scaled by $(r/3)^3$. The base cake has volume 27π, so:

$$V = 27\pi + V\left(\frac{R}{3}\right)^3 + 8V\left(\frac{r}{3}\right)^3$$

By collecting the V terms and factorising, we get that

$$V = \frac{729\pi}{27 - R^3 - 8r^3}.$$

2 Integer partitions

(a) [1 point] The partitions of 4 are:

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1.$$

(b) (i) [2 points] The distinct partitions of 7 are:

$$7, \qquad 6+1, \qquad 5+2, \qquad 4+3, \qquad 4+2+1.$$

(ii) [2 points] The odd partitions of 7 are:

$$7, \quad 5+1+1, \quad 3+3+1, \quad 3+1+1+1+1, \quad 1+1+1+1+1+1+1.$$

(c) (i) [1 point] $p(5) = 7$ – the best way to see this is just to exhaust all partitions of 5:

$$5, \qquad 4+1, \qquad 3+2, \qquad 3+1+1,$$

$$2+2+1, \qquad 2+1+1+1, \qquad 1+1+1+1+1.$$

(ii) [1 point] For each partition of n, adding one (as a separate part) results in a unique partition of $n+1$, so at least $p(n+1) \geq p(n)$. These partitions don't include the partition of one part, $n+1$, so $p(n+1) > p(n)$.

(iii) [2 points] We have $p_o(5) = p_d(5) = 3$. In fact, you only needed to check one of these, because looking ahead to (e)(ii) we later show that $p_o(n) = p_d(n)$ for all n!

In any case, the odd partitions of 5 are

$$5, \qquad 3+1+1, \qquad 1+1+1+1+1.$$

and the distinct partitions of 5 are

$$5, \qquad 4+1, \qquad 3+2.$$

(d) (i) [1 point] A partition being self-conjugate is equivalent to a partition's Ferrer diagram having reflective symmetry along the diagonal. The only self-conjugate partition of 6 is $3+2+1$ (see Figure 10).

(ii) [1 point] The only self-conjugate partition of 7 is $4+1+1+1$ (see Figure 10).

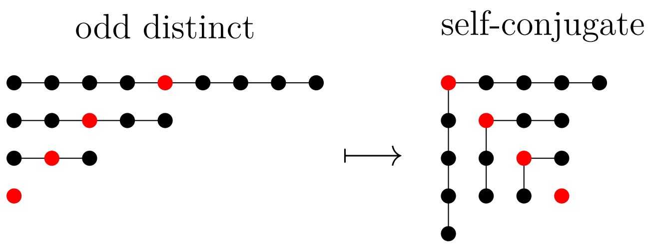
Figure 10: Ferrer diagrams of $3+2+1$ and $4+1+1+1$

(iii) [4 points] The conjugate operation creates a one-to-one correspondence between partitions (potentially mapping a partition to itself, if the partition is self-conjugate). We can also think of the conjugate as reading the partition column-wise, as opposed to row-wise (the normal way).

If the largest part of a partition is 6, the Ferrer diagram has six columns, so its conjugate will have six parts. Conversely, if a partition has six parts, the Ferrer diagram has six rows; as the Ferrer diagram descends in the size of rows, its tallest column must be 6, i.e. the conjugate has largest part equal to 6.

This establishes a one-to-one correspondence between partitions with largest part equal to 6, and partitions with six parts, hence they are equal in number for a partition of any n.

(iv) [4 points] Start with an odd distinct partition. Take the middle point in each row and use it as a "hinge" for a new Ferrer diagram:



The odd distinct partition forms a self-conjugate diagram as each hinge reflects along the diagonal. This is also a *valid Ferrer diagram* as each part in the original partition is distinct (each hinge needs to be at least 2 greater than the last, which is guaranteed by the partition being distinct)!

We can go backwards, too, as each self-conjugate partition gives us the hinges along the diagonal. It's odd because there's an equal number of points to the right of and below each hinge; it's Ferrer, and moreover distinct, because the groupings form a strictly decreasing sequence. This establishes a one-to-one correspondence between odd distinct partitions and self-conjugate partitions, hence they are equal in number for a partition of any n.

(e) (i) [5 points] Recall that we defined

$$A = (1+x)(1+x^2)(1+x^3)\cdots$$
$$B = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

We can associate the coefficient of x^n in A to $p_d(n)$, and the coefficient of x^n in B to $p_o(n)$. Why? Consider $p_d(3)$, for example. The only distinct partitions for 3 are 3 and $2+1$. Now consider the coefficient of x^n in A. We can find it by thinking what we need to multiply together to make a term of x^3; e.g., we can choose

$$(\mathbf{1}+x)(\mathbf{1}+x^2)(1+\boldsymbol{x^3})(\mathbf{1}+x^4)\cdots$$

or

$$(1+\boldsymbol{x})(1+\boldsymbol{x^2})(\mathbf{1}+x^3)(\mathbf{1}+x^4)\cdots$$

These correspond to the distinct partitions 3 and $1+2$, respectively. For example, choosing to multiply by the x^2 is the same as including 2 as a part in your partition; choosing to multiply by 1 instead of x^2 is the same as not including 2 in your partition.

This generalises to any x^n in A: its coefficient tells us the number of distinct partitions.

The same goes for B with odd partitions. The first bracket tells us the number of 1's we add to the sum; the second bracket tells us the number of 3's we add to the sum; and so on.

(ii) [5 points] Time to crunch some algebra! The hint tells us to consider $\dfrac{1-x^{2k}}{1-x^k}$. On the one hand,

$$\frac{1-x^{2k}}{1-x^k} = \frac{(1-x^k)(1+x^k)}{1-x^k} = (1+x^k)$$

so we can think of A as the infinite product

$$\begin{aligned} A &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\ &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdot\frac{1-x^{10}}{1-x^5}\cdots \end{aligned}$$

We can see that all of the numerators $(1-x^{2k})$ will eventually cancel with a denominator later on:

$$= \frac{\cancel{1-x^2}}{1-x}\cdot\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\cdot\frac{\cancel{1-x^6}}{1-x^3}\cdot\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\cdot\frac{\cancel{1-x^{10}}}{1-x^5}\cdots$$

This leaves us with A equal to

$$A = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

These are each infinite geometric sums, with first term 1 and ratio x, x^3, x^5, and so on respectively. That is,

$$\begin{aligned} &= (1 + x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots)\cdots \\ &= B. \end{aligned}$$

Remark. This was a lovely proof which Leonhard Euler came up with in 1748 – that the number of distinct partitions of n equals the number of odd partitions of n, and that this can be proved using these types of infinite products which we call *generating functions*.

3 Random tic-tac-toe

(a) [2 points] A game of tic-tac-toe ended in a draw if and only if the end grid does not contain any 3-in-a-rows of all Xs or all Os. If there was a 3-in-a-row, then at some point during the game, the robot would complete that 3-in-a-row, so that game ends with a robot winning.

(b) [3 points] Given an end grid of a game that didn't end in a draw, we cannot always determine who the winner was. If we get an end grid with a 3-in-a-row from both X and O, such as the grid shown below, then either Xeep or Obot could've won this game, depending on whose 3-in-a-row was formed first in the sequence.

X	O	X
X	O	X
X	O	O

(c) [7 points] Since all legal moves are made with equal probability, the resulting end grid of a game of random tic-tac-toe is equally likely to be any one of the 126 possible end grids. From (a), the end grid indicates a drawn game if there is no 3-in-a-row. So if we count the number of drawn end grids, then divide that by 126, that result is the proportion of drawn end grids, which is the probability that a game of random tic-tac-toe ended in a draw.

The big task is counting the number of drawn end grids, i.e. those without a 3-in-a-row. There are various systems you may use to keep track of your grids, to make sure you don't miss any cases or count the same grid twice. The system we show here is just one way to approach this.

We approach this by divide and conquer. We'll separate the end grids into smaller buckets, and then calculate the number of drawn end grids within each smaller bucket. In any end grid, there are either 0, 1, 2, 3, or 4 Os at the corners, so one way you may divide and conquer this problem is to consider cases of the number of Os at the corners.

Four Os at the corners. Recall that all end grids have 4 Os and 5 Xs. There is only 1 end grid with 4 Os at the corners, and it has a 3-in-a-row of Xs, as shown below. So there are **0** drawn end grids in this case.

O	X	O
X	X	X
O	X	O

Three Os at the corners. We draw in the corners first, shown below on the left. To avoid making a 3-in-a-row, we're forced to draw 3 more Xs between the Os as shown on the right.

O		O
O		X

O	X	O
X	X	
O		X

But now we're stuck, because we need to place one more X, which will form a 3-in-a-row. So there are in fact **0** drawn end grids in this case.

Two Os at the corners. We can't have the Os at opposite corners because then no matter our choice for the centre square, we will form a 3-in-a-row. Therefore, we'll consider arrangements with the Os at neighbouring corners.

O		X
X		O

O		O
X		X

In the grid on the right, we force the top-middle to be X and the bottom-middle to be O. We have one more O to place in the grid, and it turns out that any of the placements results in a drawn end grid!

O	X	O
X	O	X

→

O	X	O
O	X	X
X	O	X

O	X	O
X	O	X
X	O	X

O	X	O
X	X	O
X	O	X

Don't forget that we each rotate each of these grids to produce 4 distinct end grids. So in total, these are **12** drawn end grids in this case.

One O at the corners. Having 1 O and 3 Xs at the corners forces 3 more Os as shown below, then that forces the remaining squares to be Xs. This resulting grid has no 3-in-a-row, so it counts toward our total. Since this also can be rotated, there are **4** drawn end grids in this case.

O		X
X		X

O		X
	O	O
X	O	X

O	X	X
X	O	O
X	O	X

No Os at the corners. In this case, we place 4 Xs at the corners, but we need to place one more X, which cannot be done without forming a 3-in-a-row. So there are **0** drawn end grids in this case.

Any end grid falls into exactly one of the above cases, so we have accounted for all possible drawn end grids. Figure 11 shows the drawn grids we've found (note that any rotation of these grids is also a drawn grid).

O	X	O
O	X	X
X	O	X

O	X	O
X	O	X
X	O	X

O	X	O
X	X	O
X	O	X

O	X	X
X	O	O
X	O	X

Figure 11: These grids and their rotations are all the possible drawn end grids

Adding up the subtotals from each case, there are in total 16 drawn end grids. Therefore, the probability of random tic-tac-toe ending in a draw is 16/126, or 8/63.

(d) [4 points] A 3-in-a-row of Os can run either vertically, horizontally, or diagonally. But notice how if we have our 3-in-a-row of Os run vertically, that leaves two more vertical rows and only one more O left to block them, so here we will also have a vertical 3-in-a-row of Xs. The same applies to a horizontal 3-in-a-row of Xs. In order to get an end grid with a 3-in-a-row of Os, but not a 3-in-a-row of Xs, we must have the Os run diagonally. There are 2 diagonals to choose from for our 3-in-a-row of Os, and then there are 6 spots remaining to place the final O. In total, there are $2 \times 6 = 12$ such end grids.

(e) [4 points] Using the logic from (d), we now want our 3-in-a-row of Os to run either horizontally or vertically. There are 3 horizontal rows and 3 vertical rows to choose from, giving 6 options for where to place our 3-in-a-row of Os. Then there are 6 spots remaining from which to place our final O, and any choice also creates a 3-in-a-row of Xs. In total, there are $6 \times 6 = 36$ such end grids.

(f) [6 points] There is a $1/4$ probability that Obot could achieve a 3-in-a-row in 3 moves, since after 3 moves, 1 grey square will be empty, and we need that to be the bottom-right square. There is a $1/10$ probability that Xeep gets a 3-in-a-row in 3 moves, since there are 10 total pairs of white squares, and we need the pair of white squares on the right to be empty after three moves. There is a $2/5$ probability that Xeep gets a 3-in-a-row in 4 moves or fewer, since there would be 1 white square empty, and we need it to be one of the 2 white squares on the right.

If we let the game play out until the grid is filled, in the 1/4-probability case that Obot gets a 3-in-a-row in 3 moves, Xeep must hit its own 3-in-a-row in 3 moves in order to win. This case is a $\frac{1}{4} \times \frac{1}{10} = \frac{1}{40}$ probability win for Xeep. The other case is the 3/4-probability case is that Obot only gets a 3-in-a-row in exactly 4 moves. For Xeep to win it must hit a 3-in-a-row in 4 moves or fewer, so the probability for this case is $\frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$. If Xeep doesn't hit a 3-in-a-row in 4 moves or fewer then it must lose.

In total, the probability that Xeep wins ric-rac-roe is $\dfrac{1}{40} + \dfrac{3}{10} = \dfrac{13}{40}$.

(g) [4 points] To find the probability Obot wins random tic-tac-toe, consider the end grids that could result in a win for Obot. In (d), we showed there are 12 grids with a 3-in-a-row for O and not for X. If a game has this end grid, we can be certain that Obot won this game. In (e), we showed there are 36 grids with a 3-in-a-row for both O and X, and in this case we cannot be certain of who won. In (f), which shows one such ambiguous case, there is a $1 - \frac{13}{40} = \frac{27}{40}$ probability that Obot wins if we assume that the final end grid has a 3-in-a-row for both O and X. So if we are given one of the 36 end grids with a 3-in-a-row for both players, we can say with $\frac{27}{40}$ certainty that Obot won that game.

Since no other end grids have a 3-in-a-row of Os, these are the only cases which Obot could win. The probability Obot wins in random tic-tac-toe is therefore

$$\frac{12}{126} + \frac{36}{126} \cdot \frac{27}{40} = \frac{121}{420}.$$

The probabilities of Obot winning, of Xeep winning, and of a draw must add up to 1, since these are the only possible outcomes. Since from (c) the probability of a draw is $\frac{8}{63}$, the probability that Xeep wins is

$$1 - \frac{8}{63} - \frac{121}{420} = \frac{737}{1260}.$$

4 FRACTRAN

(a) [3 points] We have the following logs:

$$\begin{aligned} 4 &\to (4, 10, 25) \\ 18 &\to (18, 30, 50, 125) \\ 24 &\to (24, 40, 100, 250, 625) \end{aligned}$$

(b) [1 point] Note that doing it only once results in $2^x 3^{y-1} 5$ if $y > 0$. If $y = 0$, then it results in $2^{x-1} 5$ if $x > 0$. If both x and y are 0, then no fraction in the program will be multiplied, and as such we would have reached the end of this program.

So, this leads us to see that it first converts all the 3's into 5's, and then converts all the 2's into 5's. This means we have a program that sends $2^x 3^y$ to 5^{x+y}.

(c) [2 points] One example is simply plugging in 3, as under the first program it is sent to 2 and the second program keeps it at 3. More specifically, any positive integer with more factors of 3 than factors of 2 would work in this problem.

(d) [2 points] One example is the program $\left[\frac{1}{1}\right]$ and any positive integer n, as 1 divides all positive integers. If you want a program that runs infinitely on only certain inputs, note that $\left[\frac{k}{k}\right]$ runs infinitely on inputs of the form kn but halts on inputs of the form $kn + 1$ if $k > 1$.

(e) [5 points] We have the pairs (1, D), (2, B), (3, C), (4, E), and (5, A). Note that these proofs can be formalised with induction.

For the first program, note that it simply clears any powers of 2. The only program on the right that does this is D.

For the second program, note that for every 2 in the input, it is multiplied by 3^2, meaning that $2^x \to 3^{2x}$. So, the program that does this is B.

By now we only have A, C, and E remaining, so we only have to deal with problems with inputs that are powers of two.

So, for the third program, plugging in just 2 gives you 9. Plugging in 4 gives you the log $(4, 18, 9)$. More formally, plugging in 2^{x+1} gives you the log $(2^{x+1}, 3^2 2^x, 3^2 2^{x-1}, \ldots, 3^2)$. So, this is C.

For the fourth program, consider the log $(2^x, 2^{x-1}5, 2^{x-2}3, \ldots, 2^{x-2k}3^k)$. If x is even, then this will result in the output $3^{x/2}$. If x is odd, then we have the log end with $(2 \cdot 3^{(x-1)/2}, 3^{(x-1)/2}5, 3^{(x-1)/2})$, which tells us that this function is E.

Lastly, for the fifth program, we first compute it on 2 and get $(2, 10, 15, 9)$. Then consider the general log

$$(2^x, 2^x 5, 2^{x-1} \cdot 3 \cdot 5, 2^{x-2}3^2 5, \ldots, 2^{x-k}3^k 5, \ldots, 3^x 5, 3^{x+1})$$

So, the fifth program is A.

(f) [2 points] Derrick's program would send x to $2x$, but it would also send $2x$ to $4x$. This means that some denominator in Derrick's program must divide $2x$, but this means that it can never terminate on $2x$, meaning that when x is put in as input, it will run forever. So, Derrick's program is impossible.

(g) [3 points] Every FRACTRAN program is finite, so consider the least common multiple of all of the denominators, call it L. Note if 1 is a denominator, then the program would never terminate, so we must have that $L + 1$ is coprime to all of the denominators. Also note that this implies that L is larger than 1. So, if I plugged in $L + 1$, it would not trigger on any denominator, and as such would not be sent to 1. So, Patrick's program is impossible.

(h) [3 points] We have the program

$$\left[\frac{3}{12}, \frac{5}{6}, \frac{3}{4}, \frac{5}{2}\right].$$

Note that when we plug in 2, it is sent to 5. Then, if we plug in 2^2, it is sent to 3. If we plug in 2^3, it is sent to 5. So, we can prove that this works by induction. Assume that for $2^{2k}3$ that it is sent to 3. Now, we consider:

$$(2^{2k+2}, 2^{2k}3, 2^{2(k-1)}3, \ldots, 3)$$

So, assume that for $2^{2k-1}3$ that it is sent to 6. We have the log:

$$(2^{2k+1}, 2^{2(k-1)+1}3, 2^{2(k-2)+1}3, \ldots, 2 \cdot 3, 5)$$

Note how we had to induct on something that was not just a power of 2.

(i) [4 points] There are a few possible solutions to this, but the simplest solution is of the form

$$\left[\frac{7}{30}, \frac{7}{6}, \frac{7}{10}, \frac{7}{15}, \frac{7}{2}, \frac{7}{3}, \frac{7}{5}\right].$$

Very simply put, the first fraction sends $2^x 3^y 5^z$ to $2^{x-m}3^{y-m}5^{z-m}7^m$ where $m = \min(x, y, z)$. Note then that we have cleared at least one factor. The rest of the program simply continues this. One can prove this then via casework and induction, using the equality

$$\max(x, y, z) = m + \max(x - m, y - m, z - m).$$

(j) [5 points] The program for this question is

$$\left[\frac{385}{33}, \frac{1}{11}, \frac{3}{7}, \frac{11}{2}, \frac{1}{3}\right].$$

Note $385 = 5 \cdot 7 \cdot 11$. Consider the log

$$\begin{aligned}(2^x 3^y, 2^{x-1}3^y 11, 2^{x-1}3^{y-1}5^1 7^1 11^2, \ldots \\ 2^{x-1}5^y 7^y 11^{y+1}, \ldots, 2^{x-1}5^y 7^y, \ldots \\ 2^{x-1}3^y 5^y, \ldots \\ 2^{x-k}3^y 5^{yk}, \ldots \\ 3^y 5^{xy}, \ldots \\ 5^{xy})\end{aligned}$$

As 5 does not actually appear in the denominator, one can induct on $2^{x-1}3^y$.