Oxford Mathematics Team Challenge
Maps Round Solutions (with Questions)

Saturday, 8th March 2025

At the start of the next page is the answer key to the Maps Round, followed by solutions to each question.

Errata

Unfortunately this year's Maps Round had several inconsistencies and errors. We apologise for any confusion caused. The list of errata follows:

- **B7**'s wording was incorrect (we did not want to use the Greedy Algorithm).
- **C1**'s answer was not an integer.
- **D5** assumed knowledge of the sum of squares formula (not on the syllabus).
- **G5**'s wording was incorrect (wrong definition of breaking even).

All errors above have been amended by modifying the questions and solutions appropriately; the tests we have released online are updated accordingly.

	A	B	C	D	E	F	G
1	52	2	53	80	126	0251	11
2	10	34	5	39	101	9	7
3	50	66	63	64	135	2029	16
4	9	3	36	FREE	8	100	16
5	63	41	13	15	8	144	312
6	882	3	72	6	8	317	3
7	281	99	10	30	294	912	5050

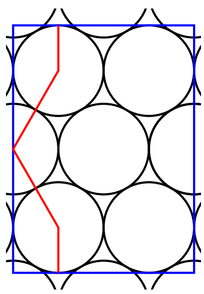
Column A

A1. [10 points] Consider a cylinder with an army of 6 ants living on its curved surface. The ants each have a territory, which is the region of points which are less than 1 cm away when distance is measured along the surface. Territories aren't allowed to cross the edges of the cylinder, and – because they don't get along – no two ants' territories can overlap.

Let P, H be its base perimeter and height of the cylinder respectively. The quantity $C = 2(P + H)$ has a minimum value of $a + b\sqrt{3}$ cm, where a, b are integers. What is the value of $a + 10b$? *[Hint: What does C represent geometrically?]*

Sol. It may be hard to visualise what the ants' territory look like, but there is a sneaky trick: we can flatten the cylinder by cutting it along its height and "unrolling" it onto a plane rectangle. Flattening the cylinder does not change the distance between any pair of points. So the ants' territories correspond to discs in a rectangle. However, we have to keep in mind that circle packings on this rectangle must loop around on one of the pairs of sides. Moreover, notice that C is equal to the perimeter of this rectangle. Intuitively, the closer the rectangle is to a square, the smaller its perimeter *(but why? Hint: complete the square.)*. So we should try to pack 6 circles into a rectangle close to a square.

Here we use another trick: recall that for a circle packing of the whole plane, the tightest arrangement is that of a hexagonal grid (where each circle touches 6 other congruent circles). We can find the optimal answer by drawing the rectangle with smallest perimeter on this grid, such that circles it contains 6 circles which can only cross its looping edges. From here, working by trial and error, we see that the following arrangement is the optimal rectangle:



This gives the minimum C_{min} to be $2(4 + (2 + 2\sqrt{3})) = 12 + 4\sqrt{3}$, so $a + 10b = \boxed{52}$.

A2. [7 points] A circle C centred at $(0,0)$ has a growing radius which starts at 0 and increases at a constant rate of 10 units per second. The line L starts at $y = -1$ and moves vertically up at a rate or r units per second. The (two) points of intersection between C and L will trace out a curve in the plane. What's the minimum value of r such that this curve is unbounded?

Sol. Solution 1: Consider that the only way for the curve to be unbounded is if the line never moves past the circle. This means some point(s) of the circle must "move" away from the origin quicker than the line.

The top point of the circle is clearly moving up the fastest among all the points of the circle. Therefore the speed of the line must be at most 10, i.e. $r \leq \boxed{10}$.

Solution 2: Let's solve for the points of intersection directly by setting up the following equations,

$$x^2 + y^2 = (10t)^2, \tag{1}$$

$$y = rt - 1, \tag{2}$$

where t denotes the time after the curves start moving.

Substituting (2) into (1) gives

$$x^2 + r^2t^2 - 2rt + 1 = 100t^2 \tag{3}$$

$$(r^2 - 100)t^2 + (-2r)t + (x^2 + 1) = 0 \tag{4}$$

If the two curves do intersect at time t, there must be solutions for (x, y) and hence for t. Since we have a quadratic equation in t, we set the discriminant to be nonnegative, i.e.

$$\begin{aligned}
(2r)^2 - 4(r^2 - 100)(x^2 + 1) &\geq 0 \\
4r^2 - 4r^2(x^2 + 1) + 400(x^2 + 1) &\geq 0 \\
(400 - 4r^2)x^2 + 400 &\geq 0 \\
(100 - r^2)x^2 + 100 &\geq 0.
\end{aligned}$$

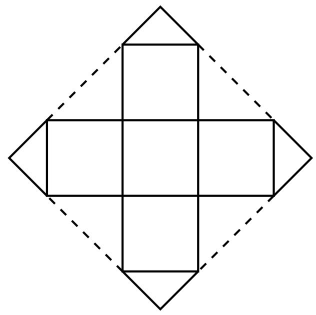
Consider cases: if $r > 10$, then this inequality forces x to be bounded, since the x^2 term has a negative coefficient. Hence the locus is bounded.

If $r \leq 10$, then for any x we know that $(100 - r^2)x^2 \geq 0$, so the inequality is always true i.e. for all values of x there exists a time t when the intersection point has x-coordinate x. This means the locus is unbounded. We conclude $r \leq \boxed{10}$.

A3. [4 points] We are trying to wrap a cube of side length 2 by folding a square sheet of paper around it (without cutting it). What's the minimum area of wrapping paper needed to fully wrap the cube?

Sol. To fold a cube from a square, we need to fit a folding net of the cube inside the square.

There are a finite number of cube nets, i.e. connected figure made of 6 squares which can be folded into a cube; the following diagram shows all possibilities. So we just pick the best net, and it's clear that we should pick the shortest diameter. The best way to fit this pattern on a square is as in the following diagram:



The outer square has a side length of $4 \times \frac{2}{\sqrt{2}}$, and hence an area of $\boxed{32}$.

A4. [3 points] The equation $15x + py = 360$ has p solutions in which x, y are non-negative integers. What is the sum of all possible values of p?

Sol. We can see an initial solution of $(x, y) = (24, 0)$ as $15 \times 24 = 360$. Since x is non-negative, it therefore can only take the values $0, \ldots, 24$, so $0 \leq p \leq 24$. It would be feasible to proceed by trial and error from here, but let's try to reason further.

From the solution $(24, 0)$, and supposing other solutions exist, we can find the "previous solution" in the following way: if we find the "smallest" non-zero integers (x_0, y_0) for which $15x_0 + py_0 = 0$ $(*)$, then

$$(15x + py) + (15x_0 + py_0) = 360 + 0$$

Rearranged, this equals

$$15(x + x_0) + p(y + y_0) = 360$$

hence generating a new solution. Rearranging $(*)$, we get $y_0 = -\frac{15}{p}x_0$.

p is either a factor of 360 or it isn't. If it isn't, then $-\frac{15}{p}x_0$ is an integer only if x_0 is a multiple of p, so the smallest non-zero value of x_0 *is* $x_0 = p$, which gives $y_0 = 15$. This means that solutions are of the form

$(24 - kp, 15k)$ for non-negative k (increasing k is repeatedly adding (x_0, y_0) our initial solution). But the smallest value of p which isn't a factor of 360 is 7, which means we can add y_0 at most $\frac{360}{7\times 15} \approx 3.4$ times, which means there are at most 4 distinct values of y, so we definitely don't have 7 solutions, let alone p.

So p – if it exists! – is a factor of 360. Therefore $15x$ always must have a factor of p, as $360 - py$ does, which limits our options of x to $1 + 360/\operatorname{lcm}\{p, 15\}$. We check when $1 + 360/\operatorname{lcm}\{p, 15\} = p$ across the possible values of p (factors of 360); this is precisely when $p = 9$. Thus the sum of values of possible p is $\boxed{9}$.

A5. [4 points] Consider the set S of fractions of the form a/b, for all integer pairs (a, b) which satisfy $0 < a < b < 15$. How many numerically distinct elements does S contain?

Sol. We can count the number of distinct elements by counting the number of fractions in lowest form (any fraction not in lowest form will have its lowest form also present in the set S).

By fixing $a = 1, 2, ..., 14$, we find the number of lowest form a/b as $13 + 6 + 8 + 5 + 8 + 3 + 6 + 3 + 4 + 2 + 3 + 1 + 1 = \boxed{63}$.

A6. [7 points] A cuboid has side lengths that are all integers in cm, and its surface area is N cm^2, and its volume is N cm^3. What is the maximum possible value of N?

Sol. Let (positive) integers x, y, z be the three side lengths of the cuboid in cm. The surface area is equal to the sum of six rectangles, and hence equals $2(xy + yz + zx)$. The volume is equal to xyz.

We are given that the surface area (measured in cm^2) and volume (measured in cm^3) are both equal to some number N; so we set up an equation

$$2(xy + yz + zx) = xyz$$

and try to solve for x, y, z.

Now dividing both sides by xyz we obtain the equation

$$\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Now WLOG $x \le y \le z$. If $x > 6$ then we have $\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{3}{6}$, a contradiction. So we must have $x \le 6$. But note we also must have

$x > 2$ as both y and z are positive integers, so we have 4 cases for x to solve for.

Now

$$2(xy + yz + zx) = xyz$$

Gives

$$2y + \frac{(2-x)yz}{x} + 2z = 0$$

Hence

$$y\left(2 + \frac{(2-x)z}{x}\right) + 2z + \frac{4x}{(2-x)} = \frac{4x}{(2-x)}$$

$$\left(y + \frac{2x}{(2-x)}\right)\left(2 + \frac{(2-x)z}{x}\right) = \frac{4x}{(2-x)}$$

$$\left(y + \frac{2x}{(2-x)}\right)\left(z + \frac{2x}{(2-x)}\right) = \frac{4x^2}{(2-x)^2}$$

If $x = 3$, we have

$$(y-6)(z-6) = 36$$

giving us the solutions $(x, y, z) =$ (3,7,42), (3,8,24), (3,9,18), (3,10,15), (3,12,12).

If $x = 4$, we have

$$(y-4)(z-4) = 16$$

giving us the solutions $(x, y, z) = (4, 5, 20), (4, 6, 12), (4, 8, 8)$.

If $x = 5$, we have

$$\left(y - \frac{10}{3}\right)\left(z - \frac{10}{3}\right) = \frac{100}{9}$$

giving us the solutions $(x, y, z) = (5, 5, 10)$.

And finally if $x = 6$, we have

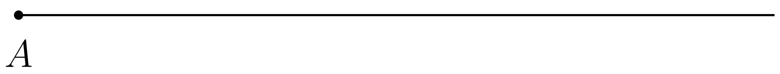
$$(y-3)(z-3) = 9$$

giving us the solutions $(x, y, z) = (6, 6, 6)$.

Solving for N, we find the maximum value N is $\boxed{882}$.

A7. [10 points] Jason's remote control car travels down a small tube with the aim of getting as far into the tube as possible. The car starts at A with an empty tank. At A there is a reserve of four litres, but the car can only hold one litre of fuel at a time. Jason can give the car the following instructions, any number of times:

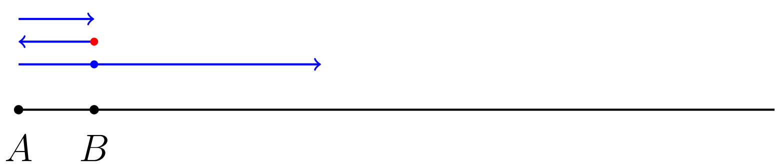
- Travel left or right, spending fuel as it moves at a ratio of 1 metre per litre;
- If the car is above a fuel reserve, it can transfer fuel between the car's tank and the reserve;
- The car can set down a fuel reserve at its current location.



The reserves that the car places down can store any amount of fuel. The maximum distance from A that Jason can reach is a/b, simplified as much as possible. What is $a + b$?

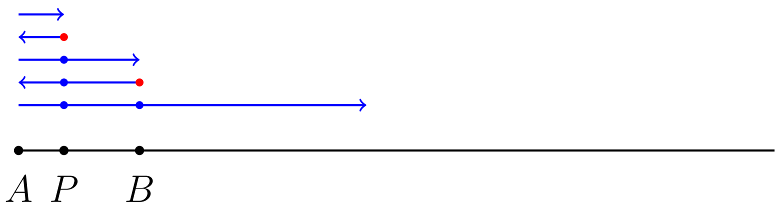
Sol. It's easier to start with less litres of fuel to try and spot a pattern. Suppose at A there is a reserve of n litres, and d_n is the furthest distance Jason's car can travel with those n litres. Clearly $d_0 = 0$ and $d_1 = 1$.

What about d_2? Jason only makes two trips, so it must be that in the first trip he's placing a reserve ahead of A with as much fuel as he can take for the second trip – call this point B and say it's at a distance x from A. B should have no more than x litres of fuel, otherwise when Jason's car reaches B on his second trip he can only take up to x litres. Therefore, on Jason's first trip, he travels x m to get to B, stores x fuel at B, and travels another x m to return to A; so $x = \frac{1}{3}$, and hence $d_2 = 1 + \frac{1}{3}$.



The red dots represent the car leaving fuel,
and the blue dots represent the car picking up fuel.

How might things work for d_3? If we place $\frac{1}{3}$ of a litre $\frac{1}{3}$ m away, then to go any further we would want to place fuel beyond B; but to do so we would need more fuel to get to some-place beyond B, which would mean taking fuel from B; but then B won't have $\frac{1}{3}$ of a litre of fuel on the last journey. What we might want to do instead is push B further along, so let's try to do just that. In the first journey, set up a point P with $\frac{3}{5}$ litres of fuel – we can put P at a distance of $\frac{1}{5}$ m from A. On the second journey, use $\frac{2}{5}$ litres to place B a distance of $\frac{1}{3}$ m from P. On the third journey, propel the car as far as you can to get $d_3 = 1 + \frac{1}{3} + \frac{1}{5}$.



The red dots represent the car leaving fuel,
and the blue dots represent the car picking up fuel.

From here there is a pattern that can be spotted – namely, that $d_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$ (showing this is quite hard!). As a simplified fraction, $d_4 = 176/105$, so $a + b = \boxed{281}$.

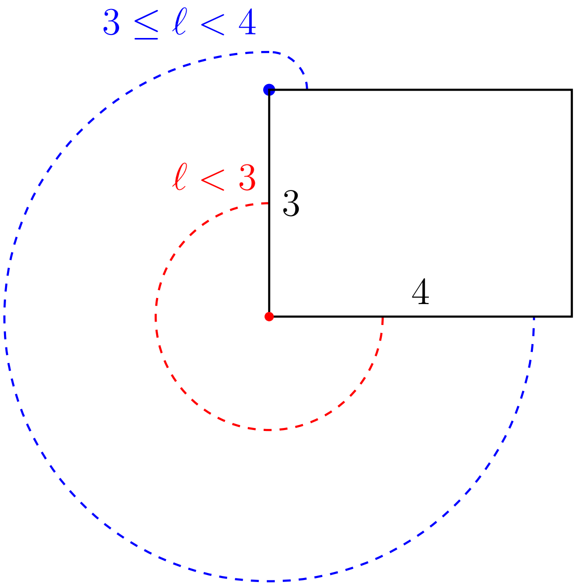
Column B

B1. [7 points] Consider an equilateral triangle ABC of side length 1. Let P be a point in the interior, and consider the region of points within ABC that is closer to P than to any of A, B, C. For different points, P, this region has different areas. The maximum area is M and the minimum area is m. What is the ratio M/m equal to?

Sol. It suffices to vary P across a sixth of the triangle ABC (e.g. between A, the midpoint of AB — call it X — and the centroid — call it G) by symmetry of ABC. Let Z_p be the area described by the point p. Construct the vectors $\mathbf{a} = AX$ and $\mathbf{b} = AG$. For the points across $\lambda\mathbf{a}$, $\lambda\mathbf{b}$ with $0 \leq \lambda \leq 1$, the area is extreme for $\lambda = 0$ or $\lambda = 1$ (between, clearly maxima and minima aren't achieved). Anywhere between these points are also not extreme. One can also verify that $Z_A < Z_X < Z_G$, so $m = Z_A$ and $M = Z_G$. Moreover m is a third of the area of the triangle, and M is two thirds, so $\frac{M}{m} = \boxed{2}$.

B2. [4 points] A goat is tied at the corner of a 3×4 shed by a rope of length ℓ, where ℓ is a rational number. How long does the rope need to be for the goat to be able to graze in an area 28π? Submit your answer as $a+b$, where $\ell = a/b$ is the simplest fraction for ℓ.

Sol. It's useful to split the goat's roaming area into cases. If $\ell \le 3$, the roaming area is just $\frac{3}{4}\pi\ell^2 \le \frac{27}{4}\pi$ – far too small. Something interesting happens for $\ell > 3$: first, if $3 < \ell \le 4$, the goat can wrap around the shorter side of the shed to get some extra area. In the diagram below, the dotted red line represents the boundary for $\ell < 3$, and the blue represents for some $3 \le \ell < 4$.



With the blue boundary, the goat can travel on the $\frac{3}{4}\pi\ell^2$ as before, but also gets an extra $\frac{1}{4}\pi(\ell-3)^2$ (you can think of the goat trying to travel around the corner, and the rope is effectively tied to the blue dot with 3 units of length less). The area encompassed by the blue boundary is $\frac{3}{4}\pi\ell^2 + \frac{1}{4}\pi(\ell-3)^2$. The maximum area in this case is $\frac{49}{4}\pi$ (taking $\ell = 4$) which isn't enough still.

If $4 \le \ell < 7$, with similar reasoning to the blue boundary the area the goat can graze in becomes $\frac{3}{4}\pi\ell^2 + \frac{1}{4}\pi(\ell-3)^2 + \frac{1}{4}\pi(\ell-4)^2$, with a maximum of 43π (with $\ell = 7$), which is enough! Set this formula equal to 28π. Then:

$$\begin{aligned} \tfrac{3}{4}\pi\ell^2 + \tfrac{1}{4}\pi(\ell-3)^2 + \tfrac{1}{4}\pi(\ell-4)^2 &= 28\pi \\ 3\ell^2 + (\ell-3)^2 + (\ell-4)^2 &= 112 \\ 5\ell^2 - 14\ell - 87 &= 0 \\ (5\ell - 29)(\ell+3) &= 0 \end{aligned}$$

Since $\ell > 0$, $\ell = 29/5$, so our final answer is $\boxed{34}$.

B3. [3 points] The following 99 people, $P_1, \ldots, P_{99}$, say the following:

$$\begin{array}{rl} P_1: & \pi = 4. \\ P_2: & P_1 \text{ is lying if and only if } P_3 \text{ is.} \\ P_3: & P_2 \text{ is lying if and only if } P_4 \text{ is.} \\ \vdots & \quad \vdots \\ P_n: & P_{n-1} \text{ is lying if and only if } P_{n+1} \text{ is.} \\ \vdots & \quad \vdots \\ P_{98}: & P_{97} \text{ is lying if and only if } P_{99} \text{ is.} \\ P_{99}: & \pi < 4. \end{array}$$

How many of them are lying?

Sol. P_1 is clearly lying, and P_{99} is clearly telling the truth. If P_2 is telling the truth, then P_3 is lying (by P_2's statement being true); so P_4 is lying (by P_3's statement being false); so P_5 is telling the truth (by P_4's statement being false); and so on. If P_2 is telling the truth, then the truth-tellers are $P_2, P_5, P_8, \ldots, P_{3n-1}, \ldots, P_{98}$. However this list doesn't include P_{99}, so P_2 must be lying. It follows that $P_3, P_6, P_9, \ldots, P_{3n}, \ldots, P_{99}$ are the truth-tellers, hence there are $\boxed{66}$ liars.

B4. [2 points] Very fortunately, the triangle ABC has area 24. D is the midpoint of AB and E is the midpoint of AC. Then let M be the midpoint of DE. What is the area of triangle CME? Express your answer as a simplified fraction.

Sol. $\triangle ADC$ has half the height of $\triangle ABC$ (considering AC as the base) and hence an area of 12. $\triangle EDC$ has its height halved again (considering DC as its base) and hence an area of 6. Again, $\triangle CME$ has half the height of $\triangle CDE = \triangle EDC$ (considering CE as the base) and hence an area of $\boxed{3}$.

B5. [3 points] We say a function f is *shrike* if it satisfies the following properties: (i) the domain and range of f is the set $\{1, 2, 3, 4, 5\}$; and (ii) $f(f(x)) = x$ for all values of x in the domain.

How many different shrike functions are there?

Sol. Notice that f is shrike precisely when the following is true: for each x either $f(x) = x$ or $f(f(x)) = x \neq f(x)$. We count the number of shrike functions by considering these two cases.

It will clarify our logic by introducing some notation: let a_n be the number of shrike functions on the set $\{1, 2, \ldots, n\}$. The original problem asks us to find a_5, but we will derive a recurrence relation.

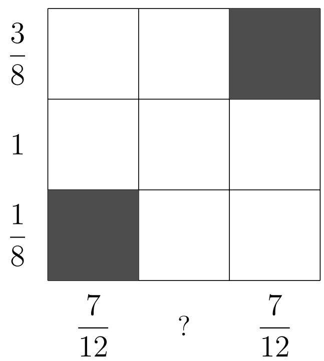
First suppose $f(1) = 1$. This doesn't restrict the values for $2 \leq x \leq n$, so this case gives us a_{n-1} functions.

If on the contrary we have $f(1) = y \neq 1$ (of which there are $n-1$ choices for y), then we must have $f(y) = f(f(1)) = 1$. These two elements are fixed, but everything else is free. This case gives us $(n-1)a_{n-2}$ functions.

Since these are the only possible cases, we get the identity $a_n = a_{n-1} + (n-1)a_{n-2}$ for all n. To obtain a_5, we can work backwards from smaller values of n. Check that $a_1 = 1$ and $a_2 = 2$; then we get $a_3 = 2 + 2 \cdot 1 = 4$, $a_4 = 4 + 3 \cdot 2 = 10$, $a_5 = 10 + 4 \cdot 4 = \boxed{26}$.

B6. [4 points] With the unit fractions

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \frac{1}{24},$$



Angus arranges them in the white squares in the grid below, and then writes the sums of the rows and columns as shown in the diagram. What's the reciprocal of the sum of the middle column?

Sol. Filled in, the grid looks like

$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
$\frac{1}{8}$		$\frac{1}{24}$	$\frac{1}{12}$
	$\frac{7}{12}$	?	$\frac{7}{12}$

So ? $= \frac{1}{3}$, so its reciprocal is $\boxed{3}$. To get started on filling in the grid, it's helpful to note that 1/8 must be the sum of 1/12 and 1/24 (all the other fractions are too big); this can then help with the right-hand 7/12 column, which in turn helps narrows down the possibilities for the left-hand 7/12 column.

B7. [7 points] E_1 starts with the fraction $\frac{1}{2}$. E_2 adds a unit fraction to E_1's fraction to get $\frac{2}{3}$. E_3 adds a unit fraction to E_2's fraction to get $\frac{3}{4}$. This process keeps going until E_n adds $\frac{1}{9900}$. What is n?

Sol. Let's try to spot a pattern. We can say E_1 adds $\frac{1}{2}$. E_2 adds $\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$, and E_3 adds $\frac{3}{4}-\frac{2}{3}=\frac{1}{12}$. This leads us to form the equation

$$\frac{n}{n+1}-\frac{n-1}{n}=\frac{1}{9900}$$

After some algebra, we get $n^2+n-9900=0$, which factorises to $(n-99)(n+100)=0$, so $n=\boxed{99}$.

Alternatively, you could've noticed that $2, 6, 12, \ldots$ are twice the triangle numbers $1, 3, 6, \ldots$, so equivalently we want to know which double triangle number 9900 is. That'll form the same equation as above.

Column C

C1. [4 points] Three circles X, Y, Z having centres A, B, C respectively are externally tangent to each other. Let D on AB, E on BC, F on CA be the intersection points of each pair of circles. Let T_D, T_E, T_F be the circles' tangent lines at points D, E, F respectively.

Suppose the lines T_D, T_E form an angle $90°$, the lines T_E, T_F form an angle $120°$, the lines T_F, T_D form an angle $150°$, and $AB=5$. Then $BC=\frac{a\sqrt{3}}{b}$ for some positive integers a, b. What is $10a+b$?

Sol. First we solve for each angle of $\triangle ABC$: Draw the lines AB, BC, CA. Notice that the lines AB, T_D, T_F, AC bound a quadrilateral Q_A.

Since T_D is the tangent line to circles X and Y, it must be perpendicular to the radius (vector) of each circle, i.e. T_D is perpendicular to AD and DB. Therefore the angles of Q_A at D and F are right angles.

Recall that angles in a quadrilateral sum to $360°$, which gives us $\angle BAC = 360° - 90° - 90° - \angle(T_D, T_F) = 30°$.

Similarly we obtain $\angle ABC = 90°$ and $\angle BCA = 60°$. (TIP: check that these three angles really add to $180°$!)

Now that we have all angles, use the sine formula:

$$\frac{AB}{\sin \angle BCA} = \frac{BC}{\sin \angle BAC}$$
$$\frac{5}{\sqrt{3}/2} = \frac{BC}{1/2}$$
$$BC = \frac{5\sqrt{3}}{3}.$$

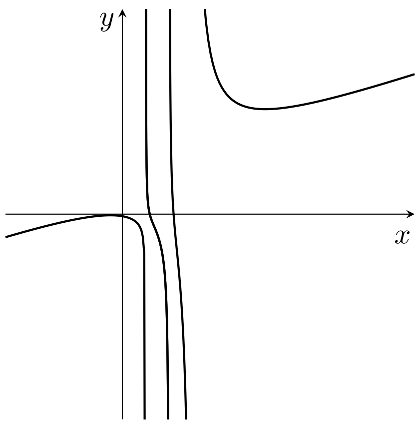
It follows that $10a + b = \boxed{53}$.

C2. [3 points] Let $a \neq 0$. How many segments does the curve

$$\frac{1}{x-a} + \frac{x}{x-2a} + \frac{x^2}{x-3a}$$

split the 2D plane into?

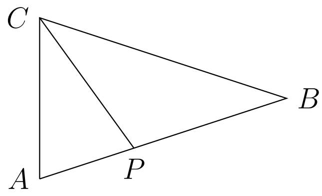
Sol. The expression goes to ∞ as $x \to \infty$, and to $-\infty$ as $x \to -\infty$. Further, there are vertical asymptotes at $a, 2a, 3a$, which are all distinct since $a \neq 0$. This divides the plane into $\boxed{5}$ regions as can be seen in a graph.



C3. [2 points] There is only one integer n between 1 and 100 such that the sum of the digits of n is half of the sum of the digits of $3n$. What is n?

Sol. For brevity, denote $S(n)$ denote the sum of the digits of n. We can write $2S(n) = S(3n)$. Note $S(3n)$ is a multiple of 3, so $S(n)$ is a multiple of 3, so n is a multiple of 3, so $S(3n)$ is a multiple of 9, so n is a multiple of 9. With this insight, our search of choices is reduced greatly, so only the multiples of 9 needing to be checked. We can then see that 63 is the only such number with this property, so the answer is $\boxed{63}$.

C4. [1 point] In the diagram, $AB = BC$, $AC = BP$, $BP = CP$ and $\angle BPC \neq \angle BAC$. What is the angle $\angle ABC$ in degrees?



Sol. Let's go angle-chasing! Say $\angle ABC = x^\circ$. Since CPB is isosceles, $\angle PCB = x^\circ$ so $\angle BPC = 180 - 2x^\circ$. Since the angles around P add to 180°, $\angle APC = 2x^\circ$. Since ACP is isosceles, $\angle PAC = 2x^\circ$, so $\angle ACP = 180 - 4x^\circ$. Now $\angle ACB = \angle ACP + \angle BCP = 180 - 3x$. Since ABC is isosceles, $\angle ACB = \angle CAB$ so $180 - 3x = 2x$, which solves to $x = \boxed{36^\circ}$.

C5. [2 points] When Luci rationalises

$$\frac{2}{\sqrt[3]{3} - 1}$$

she gets $\sqrt[3]{a} + \sqrt[3]{b} + c$, where a, b, c are positive integers. What is the value of $a + b + c$?

Sol. Let's set up our equation:

$$\frac{2}{\sqrt[3]{3} - 1} = \sqrt[3]{a} + \sqrt[3]{b} + c$$

Multiplying up,

$$\begin{aligned} 2 &= (\sqrt[3]{3} - 1)(\sqrt[3]{a} + \sqrt[3]{b} + c) \\ \implies 2 &= \sqrt[3]{3a} + \sqrt[3]{3b} + c\sqrt[3]{3} - \sqrt[3]{a} - \sqrt[3]{b} - c \end{aligned}$$

This looks a bit icky, but we can work with this. Importantly, c is positive so $-c$ is negative; for the right-hand side to be equal to a positive integer, at least one of the surds must be an integer! It won't be $c\sqrt[3]{3}$, which leaves either $\sqrt[3]{3a}$ or $\sqrt[3]{a}$ (or b, but it doesn't really matter which we focus on).

We should try small values of a which give us integers; our candidates are either $a = 8$ (since $\sqrt[3]{a}$ would equal 2) or $a = 9$ (since $\sqrt[3]{3a}$ would equal 3). One can spot that $a = 9$ leads to $b = 3$ (which cancels $\sqrt[3]{3b}$ with $\sqrt[3]{a}$) and $c = 1$, so $a + b + c = \boxed{13}$.

C6. [3 points] How many factors of 12! are odd?

Sol. The prime factorisation of 12! is $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$.

For each factor of 12!, each prime factor can be present with powers ranging from 0 to its power in the above factorization. For example, the factor 5 can be present as $5^0, 5^1$ or 5^2 – thus it provides 3 choices. Each prime number offers such choices independently, so we should multiply the number of choices together by the product rule of counting (but we should ignore choices of 2 since we only want odd factors). Thus, the number of odd factors is $6 \cdot 3 \cdot 2 \cdot 2 = \boxed{72}$.

C7. [4 points] The unit fractions $1/a$, $1/b$ and $1/c$ sum to 1, where a, b, c are positive integers. How many different solutions of (a, b, c) are there? *[Note: (a, b, c) is a different solution to, e.g., (b, a, c).]*

Sol. Without loss of generality, suppose $a \leq b \leq c$. We can spot the solutions $(a, b, c) = (2, 3, 6)$ and $(2, 4, 4)$. Suppose for a contradiction that $a = 2$ and $b > 4$; then $\frac{1}{c} = 1 - \frac{1}{a} - \frac{1}{b} > \frac{1}{2} - \frac{1}{4}$. So $\frac{1}{c} > 1/4$ so $c < 4$: a contradiction (by ordering of b, c). If $a = 3$, we may also spot $(a, b, c) = (3, 3, 3)$. One can make a similar argument that, if $a = 3$ and $b > 3$, we get no well-ordered solutions. Lastly, if $a > 3$ then since $a \leq b \leq c$ the sum can only be as big as $\frac{3}{4}$, which is not big enough. Now, removing the well-ordering restriction we get $\boxed{10}$ solutions.

Column D

D1. [3 points] Johnny picks six points in the 2-dimensional plane. He needs to draw at least N straight lines such that any pair of two points is connected by a line. What is the sum of all possible values of N?

Sol. Notice that the max is $N = 15$, achieved when no three points are colinear. When you have 3 colinear points, you collapse 3 lines into 1, reducing N by 2. When you have 4 colinear points, N reduces by 5, 5 colinear points reduces by 9, 6 colinear points reduce by 14. Considering which combinations of colinear points we can have gives us an easy way to find all possible N. Such possible values for N are 15, 13, 11, 10, 9, 8, 7, 6, 1. Giving us the answer $\boxed{80}$.

D2. [2 points] How many integer side length triangles are there where two of the sides are length 20 and 25?

Sol. Let θ be the angle between the sides with length 20 and 25 and c be the final side length of the triangle (opposite of θ). Then by the cosine rule, $c = \sqrt{20^2 + 25^2 - 2 \cdot 20 \cdot 25 \cos(\theta)} = \sqrt{1025 - 1000 \cos(\theta)}$. Now $-1 < \cos(\theta) < 1$ as $0° < \theta < 180°$ so we know that every value of $5 < c < 45$ is achieved. As we need c to be an integer, the number of possible values of c is $\boxed{39}$.

D3. [1 point] What is the sum of the coefficients on the polynomial

$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})(1 + x^{32})$$

including the constant term?

Sol. If we try expanding with the smaller powers first, note

$$(1 + x)(1 + x^2) = 1 + x + x^2 + x^3$$

and

$$(1 + x + x^2 + x^3)(1 + x^4) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$$

This (correctly) indicates that the expression is

$$1 + x + x^2 + \cdots + x^{63}$$

which means the sum of the coefficients is $\boxed{64}$ (one for each power of x). Similarly to 2. (e) of Lock-in, we can reason this expansion without brute force by thinking about the terms in each bracket I multiply, and which ones I can multiply to get each power from x^0 to x^{63}.

D4. [0 points] Free space! You don't need to submit anything.

D5. [1 point] In Agniv's collection, he has 1 lot of 1, 2 lots of 2, ..., and 10 lots of 10. What is the median of Agniv's collection?

Sol. Since there are $\frac{10 \cdot 11}{2} = 55$ elements, the median is the 28th element in ascending order, which turns out to be $\boxed{7}$.

D6. [2 points] What is the value of the smallest positive integer n with the property that there exist digits A, B, and C such that A is non-zero and $n \times AB = ACB$?

Sol. Suppose for a contradiction that $n \leq 5$ and $n \times AB = ACB$. Then we know that

$$100A \leq ACB = n \times AB \leq 5 \times AB = 50A + 5B$$

But this means that $10A \leq B$, which is a contradiction as A is non-zero and $0 \leq B \leq 9$. $n = 6$ is possible as $n \times 18 = 108$. So the smallest n is $\boxed{6}$.

D7. [3 points] Peter has a straight strip of paper 1 cm wide and lays another straight strip 2 cm wide overlapping it. The resulting overlapping region of the two strips of paper has area 4 cm^2. What is the angle between the two strips of paper?

Sol. If we let θ be the angle between the two strips, then by the sine area rule, we get $2 \cdot 1 \cdot \sin\theta = 4$. Thus $\theta = \boxed{30^\circ}$.

Column E

E1. [4 points] Rebekah chooses 5 random distinct non-zero digits a_1, a_2, a_3, a_4, a_5 and computes the product $(a_1^{a_1} - 1)(a_2^{a_2} - 1)(a_3^{a_3} - 1)(a_4^{a_4} - 1)(a_5^{a_5} - 1)$. Let p be the probability that the last digit of the product is 0. What is $1/p$?

Sol. By the pigeonhole principle, there must be one of a_i which is odd, and thus $a_i^{a_i} - 1$ is even. So for the last digit of the product to be 0, we need one of a_i to be such that $(a_i^{a_i} - 1)$ has a factor of 5. The only a_i

that satisfy this are $a_i = 1, 4, 6, 8$ thus the only way to NOT have the last digit of the product be 0 is if $(a_1, a_2, a_3, a_4, a_5)$ is some permutation of $(2, 3, 5, 7, 9)$. Thus $p = \frac{1}{126}$ so the answer is $\boxed{126}$.

E2. [3 points] A sequence is defined by $u_1 = a$, $u_2 = b$,

$$u_{n+1} = \begin{cases} u_{n-1} + u_n & \text{if } n \text{ is odd} \\ u_{n-1} - u_n & \text{if } n \text{ is even} \end{cases}$$

If the 100th and 101st terms are equal, what is the value of $100 - a/b$?

Sol. Writing out the first few terms of the sequence:

$$a, b, a+b, -a, b, -a-b, -a, -b, -a-b, a, -b, a+b, a, b$$

Since the a, b at the end falls on $n = 13, 14$ respectively, the pattern continues. One can see that the 97th and 98th term are a, b, and hence the 100th and 101st term are $-a, b$. Thus $100 - a/b = \boxed{101}$.

E3. [2 points] The triangle PQR satisfies $PQ = PR$. Let S be a point on PQ and T be a point on QR such that $\angle QTS = 90°$. Given that $QT = 20$, that $QS = 25$, and that the area of QST is two ninths the area of PQR, what is the perimeter of PQR?

Sol. Let Q' denote the reflection of the point Q over T. Then we know that the area of QSQ' is $\frac{4}{9}$ the area of PQR. But we also know that QSQ' is similar to QPR and thus the side lengths of QSQ' are $\sqrt{\frac{4}{9}} = \frac{2}{3}$ the side lengths of QPR. And thus as the perimeter of $QSQ' = 90$, the perimeter of PQR is $90 \cdot \frac{3}{2} = \boxed{135}$.

E4. [1 point] Find the value of $2+\sqrt{20 + \sqrt{202 + \sqrt{2025}}}$ rounded to the nearest integer.

Sol. $2 + \sqrt{20 + \sqrt{202 + \sqrt{2025}}} = 2 + \sqrt{20 + \sqrt{247}}$. We have the bound $15 < \sqrt{247} < 16$ so $2+\sqrt{35} < 2+\sqrt{20 + \sqrt{247}} < 2+\sqrt{36} < 8$. Now $5.5^2 = 30.25$ so $\sqrt{35} > 5.5$ thus $7.5 < \sqrt{20 + \sqrt{247}} < 8$ so the value rounded to the nearest integer is $\boxed{8}$.

E5. [2 points] A circle is drawn in the plane, and has diameter with endpoints at $P = (2, 8)$ and $Q = (8, 16)$. What is the shortest distance from the origin to a point on the circle?

Sol. The circle has centre at $(5, 12)$. So radius of the circle is

$$\sqrt{(8-5)^2 + (16-12)^2} = 5$$

The distance from the centre of the circle to the origin is $\sqrt{12^2 + 5^2} = 13$, thus the shortest distance from the origin to a point on the circle is $13 - 5 = \boxed{8}$.

E6. [3 points] The graph of $\ln x$ is stretched vertically by a factor b and results in the function $f(x)$. Given that $8^{f(x)} = x^2$ for all x, what is the value of $e^{2/b}$?

Sol. A stretch by b implies that the new function is $f(x) = b \ln x$. Taking natural logarithms on our equation,

$$\begin{aligned} x^2 = 8^{f(x)} &\implies \ln(x^2) = \ln\left(8^{f(x)}\right) \\ &\implies 2 \ln x = f(x) \ln 8 \\ &\implies \frac{2 \ln x}{\ln 8} = b \ln x \\ &\implies b = \frac{2}{\ln 8} \end{aligned}$$

Thus $e^{2/b} = e^{\ln 8} = \boxed{8}$.

E7. [4 points] A group of 99 people is such that every pair of people have exactly one friend in common. Each person in this group has a *friendship number*, which is just the number of friends that they have. What is the sum of the friendship numbers of people in this group?

Sol. One way to construct such a group of people is to have a "universal friend" who is friends with everybody. The other 98 people pair up to form a triangle with this universal friend. It turns out that this is the only way to construct such a group, but proving this is very difficult! The universal friend has friendship number 98, and the other 98 people have friendship number 2. This gives $\boxed{294}$.

Column F

F1. [7 points] What, from left to right, are the last 4 digits of 11^{2025}?

Sol. Consider the Binomial expansion of $(10+1)^{2025}$. All of the terms with 10^4 or higher are essentially irrelevant in how they affect the last four digits, so we only need to consider the last four digits of

$$^{2025}C_3 10^3 1^{2022} + {}^{2025}C_2 10^2 1^{2023} + {}^{2025}C_1 10^1 1^{2024} + 1$$

We know $^{2025}C_1 = 2025$, we can calculate

$$^{2025}C_2 = \frac{2025!}{2!2023!} = \frac{2025 \times 2024}{2} = 2025 \times 1012 = 2049300$$

In fact, it suffices to calculate 25×12 because

$$2025 \times 1012 = (2000+25)(1000+12)$$

and the terms with factors of 1000 don't affect the last four digits (since we're multiplying this term by 10^2). For $^{2025}C_3$, as long as we know there's a factor of 100, the 10^3 means this has no effect. Indeed,

$$^{2025}C_3 = \frac{2025!}{3!2022!} = \frac{2025 \times 2024 \times 2023}{6} = 675 \times 1012 \times 2023$$

675 has a factor of 25, and 1012 has a factor of 4, so we have a factor of 100.

Therefore, the only terms affecting the last four digits is 1 and 20250, so the last four digits are $\boxed{0251}$.

F2. [4 points] The digits of 3^{2025} are added up to make a new number. The digits of this new number are added up again to get a third number. We continue this process until we get a single digit number. What is the number?

Sol. The sum of digits of any number divisible by 9, is also divisible by 9. Since 3^{2025} is divisible by 9, so will its sum of digits (the new number), and hence so will also the third number. Since the sum of a number bigger than 10 is less than the number, this continues till we reach a single digit, and hence the answer is $\boxed{9}$.

F3. [3 points] Rosie, Angie and Ella are standing in a line, with Rosie being at the front and Ella at the back. Each of them are wearing a jersey with a distinct natural number picked from the set $\{1, 8, 3, 100, 2025\}$ on their back, which they cannot see themselves. Each person can see the numbers of all those ahead of them.

First, Ella says that she doesn't know if her number is even or odd. Then Angie says she doesn't know if his number is even or odd. Then Rosie then says she knows whether her number is even or odd.

What are the sum of the numbers Rosie could be wearing?

Sol. If Angie and Rosie had both the even numbers on their backs, then Ella would've known that her number was odd. Thus at least one of Angie and Rosie has an odd number on their back.

Similarly, if Rosie had an even number on her back, then Angie would've known that she had an odd number on her back, so Rosie must have an odd number on her back.

Thus the sum of possible numbers $= 1 + 3 + 2025 = \boxed{2029}$.

F4. [2 points] Let $f(x) = \sin x - \cos^2 x + \sin^3 x - \cos^4 x + \cdots + \sin^{99} x - \cos^{100} x$. Let M be the maximum value of f, and m the minimum value. What is $M - m$?

Sol. Note $f(0°) = -50$ and $f(90°) = 50$. For any other values, either $|\sin x| < 1$, in which case the subsequent sine terms become negligible, or $|\cos x| < 1$, in which case the subsequent cosine terms become negligible. There is therefore no way for $|f(x)| > 50$; so $M - m = \boxed{100}$.

F5. [3 points] Let any natural number n which satisfies

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{24}$$

be called a *binky* number. What's the smallest binky number?

Sol. Taking the reciprocal, we get

$$\frac{1}{\sqrt{n+1} - \sqrt{n}} > 24$$

Can we rationalise this? If we multiply top and bottom of the left-hand side by $(\sqrt{n+1} + \sqrt{n})$, we get

$$\sqrt{n+1} + \sqrt{n} > 24$$

Now $\sqrt{144}+\sqrt{144}=24$, so $\sqrt{144}+\sqrt{143}<24$ and $\sqrt{145}+\sqrt{144}>24$. Therefore the smallest binky number is $\boxed{144}$.

F6. [4 points] A friendly enemy writes the numbers $1, 2, 3, 4, 5$ onto a dark blue wall. Repeatedly, you erase two numbers – call them a and b, with $a > b$ – and write either $2a + b$ or $a + 7b$ onto the wall. This process repeats until there is only one number left. What is the largest number you can create?

Sol. There's a naïve approach, where we take the biggest two numbers at each available opportunity. This gives

$$(5,4) \mapsto 33; \quad (33,3) \mapsto 69; \quad (69,2) \mapsto 140; \quad (140,1) \mapsto 281.$$

But we can do better by trying to take advantage of the $a+7b$ option. If we get two medium-sized numbers then $a + 7b$ can be quite big! Indeed, the approach

$$(3,4) \mapsto 25; \quad (2,5) \mapsto 19; \quad (19,25) \mapsto 158; \quad (158,1) \mapsto 317$$

is optimal, so the largest number you can create is $\boxed{317}$.

F7. [7 points] Player 67 and Player 456 are playing a game.

They place points on a circle of radius $a > 0$ on each turn. They take turns playing, with Player 67 going first placing a point anywhere on the circle. Thereafter, each player must place a point at distance more than 2π away (as measured on the circle) from all other points. The player who is unable to place a valid point loses. One of these players with number n is guaranteed to win as long as the circle has radius bigger than r. Find $n \times r$.

Sol. For a circle with $a \leq 2$, Player 67 automatically wins, since no point is more than half the circumference ($\leq 2\pi$) away from the first point placed.

For $a > 2$, Player 456 can always place points diametrically opposite to the points placed by Player 67 in the previous turn. By symmetry, if Player 67's point is valid, Player 456's point must be. This continues until Player 67 loses.

Thus the answer is $2 \cdot 456 = \boxed{912}$.

Column G

G1. [10 points] Let $d(n)$ be the sum of the digits of n. Given that n equals $2025 \times d(n) \times d(d(n))$, how many prime factors does n have?

Sol. Note that if 9 divides n, then 9 divides the sum of its digits $d(n)$. To show this, suppose that the ones digit of n is a_0, its tens digit is a_1, and so on, so that $n = a_0 + 10a_1 + ... + 10^k a_k$. Then $d(n) = a_0 + a_1 + ... + a_k$, so $n - d(n) = (1-1)a_0 + (10-1)a_1 + ... + (10^k - 1)a_k$. Since each of these terms are a multiple of 9, their sum must be a multiple of 9 too. But if 9 divides n and 9 also divides $n - d(n)$, then it must necessarily be true that 9 divides $d(n)$ as well.

Now, since 9 divides 2025 and 2025 divides n, 9 divides n. Thus, 9 divides $d(n)$ and by extension 9 also divides $d(d(n))$. Thus, we know that $2025 \cdot 9 \cdot 9$ divides n.

Directly checking, we note that $n = 2025 \cdot 9 \cdot 9 \cdot 2$ or $n = 2025 \cdot 9 \cdot 9 \cdot 3$ each case giving n a total of $\boxed{11}$ prime factors.

G2. [7 points] The quadratic $2x^2 - 4x + 1$ has the roots α and β, with $\alpha > \beta$. The 2025th digit after the decimal point of α is 2. What is the 2025th digit after the decimal point of β?

Sol. The roots of this equation are $1 \pm \frac{\sqrt{2}}{2}$. Importantly, $2 > \alpha > 1 > \beta > 0$, and $\alpha + \beta = 2$. Considering decimal expansions, let $\alpha = 0.a_1a_2a_3\ldots$ and $\beta = b_1b_2b_3\ldots$. Then

$$\alpha + \beta = 0.a_1a_2a_3\cdots + 0.b_1b_2b_3\cdots = 1.999\ldots$$

so for each i, $a_i + b_i = 9$. Given that $a_{2025} = 2$ (which is true, I checked!), it follows that $b_{2025} = \boxed{7}$.

G3. [4 points] When 25! is multiplied by 5^n, it has the highest possible number of zero digits at the end of the number. What is the least value of n?

Sol. To solve this problem, we count the highest power of 2 that divides 25!. This is given by

$$\left\lfloor \frac{25}{2} \right\rfloor + \left\lfloor \frac{25}{4} \right\rfloor + \left\lfloor \frac{25}{8} \right\rfloor + \left\lfloor \frac{25}{16} \right\rfloor = 22$$

where $\lfloor x \rfloor$ is x rounded down to the nearest integer. We also count the highest power of 5 that divides 25! which is given by

$$\left\lfloor \frac{25}{5} \right\rfloor + \left\lfloor \frac{25}{25} \right\rfloor = 6$$

To get a zero at the end of a number, we need a factor of $10 = 2 \cdot 5$; so the highest possible number of zero digits at the end of $25! \cdot 5^n$ is 22. As we already have six 5's in 25!, the least value of n is $\boxed{16}$.

G4. [3 points] Given that $z + \sqrt{3} = \sqrt{7 + 4\sqrt{3}}$, what is the value of z^4?

Sol. One can spot that $7 + 4\sqrt{3} = 4 + 2 \cdot 2\sqrt{3} + 3 = (2 + \sqrt{3})^2$, so in fact $z = 2$. It follows that $z^4 = \boxed{16}$.

G5. [4 points] Yoshi starts with a capital of £150. A game consists of a sequence of up to fifty **A**'s and **B**'s, where **A** and **B** are the following actions:

A: Yoshi loses £1.

B: If Yoshi's capital is even, he wins £3. Otherwise, he loses £5.

Yoshi breaks even after a sequence of games if his capital is £150. What is the sum of all game-lengths where Yoshi can break even?

Sol. Considering the capital's remainder when divided by 4, after either action of **A** or **B** it always decreases by one. Thus one can only break even on games of lengths which are multiples of 4. But which such games? It is not too hard to check that one can break even on a game of length 4 (e.g., **BAAA**) so we can break even on any game length which is a multiple of 4 less than 50 (by repeating **BAAA**, for example) so we should add $4 + 8 + 12 + \cdots + 48$. This gives $\boxed{312}$.

G6. [7 points] Healy and Dobby are playing 'The Game of 2025', in which they each pick positive numbers, successively adding them to a total. The first player to reach a number at least as big as 2025 loses.

They can pick numbers to add from $\{1, 2, ..., n\}$. If Healy begins, for how many $1 \leq n \leq 7$ does Dobby always have a winning strategy?

Sol. If 2024 is divisible by $(n+1)$, then whatever is added by Healy, Dobby can always add a number so that the total is a multiple of $n+1$. At the end, Dobby will reach 2024, forcing Healy to lose.

For other n, Healy can start with a number such that, effectively, the game is restarted with a modified total a to reach, such that $a-1$ is divisible by $n+1$, with Dobby starting. Utilising the same strategy as above, Healy can win.

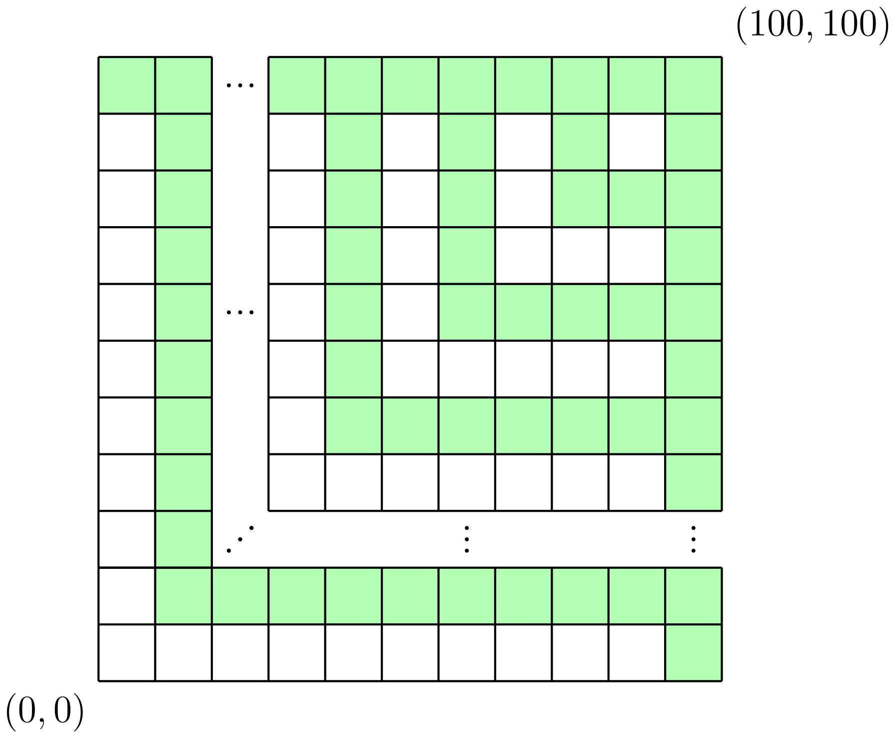
For example, if $n = 6$, Healy can add 1 so that the game becomes a 'Game of 2024', with Dobby starting, and $2024 - 1 = 2023$ is divisible by $6 + 1 = 7$. Then whatever Dobby plays, Healy can make the total a multiple of 7. At the end, Healy will reach 2023 of the modified game, which is 2024 of the original game, forcing Dobby to lose.

Thus the only n for which Dobby has a winning strategy are those such that $n+1$ divides 2024. $n = 1, 3, 7$ are the only such n in the set. Hence, the answer is $\boxed{3}$.

G7. [10 points] Ritter and his evil enemy play a game where a point P starts in a 100×100 cm^2 grid. On each turn, each player moves the point x units right and y units up, where $0 \leq x \leq 1$ and $0 \leq y \leq 1$, with at least one of x, y equal to 1. A player wins once they move the point outside the square. Ritter moves first.

Find the area of the range of starting points in the grid for which Ritter has a winning strategy.

Sol. The winning spaces are shaded below:



It's easier to calculate that the losing area is $1+5+9+\cdots+197 = 4950$, and hence the winning area is $100^2 - 4950 = \boxed{5050}$.