Oxford Mathematics Team Challenge
Maps Round Solutions

Saturday, 8th March 2025

At the start of the next page is the answer key to the Maps Round, followed by solutions to each question.

Errata

Unfortunately this year's Maps Round had several inconsistencies and errors. We apologise for any confusion caused. The list of errata follows:

- **B7**'s wording was incorrect (we did not want to use the Greedy Algorithm).
- **C1**'s answer was not an integer.
- **D5** assumed knowledge of the sum of squares formula (not on the syllabus).
- **G5**'s wording was incorrect (wrong definition of breaking even).

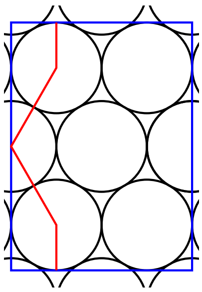
All errors above have been amended by modifying the questions and solutions appropriately; the tests we have released online are updated accordingly.

	A	B	C	D	E	F	G
1	52	2	53	80	126	0251	11
2	10	34	5	39	101	9	7
3	50	66	63	64	135	2029	16
4	9	3	36	FREE	8	100	16
5	63	41	13	15	8	144	312
6	882	3	72	6	8	317	3
7	281	99	10	30	294	912	5050

Column A

A1. [10 points] It may be hard to visualise what the ants' territory look like, but there is a sneaky trick: we can flatten the cylinder by cutting it along its height and "unrolling" it onto a plane rectangle. Flattening the cylinder does not change the distance between any pair of points. So the ants' territories correspond to discs in a rectangle. However, we have to keep in mind that circle packings on this rectangle must loop around on one of the pairs of sides. Moreover, notice that C is equal to the perimeter of this rectangle. Intuitively, the closer the rectangle is to a square, the smaller its perimeter *(but why? Hint: complete the square.)*. So we should try to pack 6 circles into a rectangle close to a square.

Here we use another trick: recall that for a circle packing of the whole plane, the tightest arrangement is that of a hexagonal grid (where each circle touches 6 other congruent circles). We can find the optimal answer by drawing the rectangle with smallest perimeter on this grid, such that circles it contains 6 circles which can only cross its looping edges. From here, working by trial and error, we see that the following arrangement is the optimal rectangle:



This gives the minimum C_{min} to be $2(4 + (2 + 2\sqrt{3})) = 12 + 4\sqrt{3}$, so $a + 10b = \boxed{52}$.

A2. [7 points] Solution 1: Consider that the only way for the curve to be unbounded is if the line never moves past the circle. This means some point(s) of the circle must "move" away from the origin quicker than the line.

The top point of the circle is clearly moving up the fastest among all the points of the circle. Therefore the speed of the line must be at most 10, i.e. $r \leq \boxed{10}$.

Solution 2: Let's solve for the points of intersection directly by setting up the following equations,

$$x^2 + y^2 = (10t)^2, \tag{1}$$
$$y = rt - 1, \tag{2}$$

where t denotes the time after the curves start moving.

Substituting (2) into (1) gives

$$x^2 + r^2t^2 - 2rt + 1 = 100t^2 \tag{3}$$
$$(r^2 - 100)t^2 + (-2r)t + (x^2 + 1) = 0 \tag{4}$$

If the two curves do intersect at time t, there must be solutions for (x, y) and hence for t. Since we have a quadratic equation in t, we set the discriminant to be nonnegative, i.e.

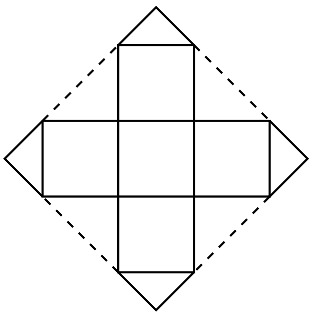
$$\begin{aligned}(2r)^2 - 4(r^2 - 100)(x^2 + 1) &\geq 0\\ 4r^2 - 4r^2(x^2 + 1) + 400(x^2 + 1) &\geq 0\\ (400 - 4r^2)x^2 + 400 &\geq 0\\ (100 - r^2)x^2 + 100 &\geq 0.\end{aligned}$$

Consider cases: if $r > 10$, then this inequality forces x to be bounded, since the x^2 term has a negative coefficient. Hence the locus is bounded.

If $r \leq 10$, then for any x we know that $(100-r^2)x^2 \geq 0$, so the inequality is always true i.e. for all values of x there exists a time t when the intersection point has x-coordinate x. This means the locus is unbounded. We conclude $r \leq \boxed{10}$.

A3. [4 points] To fold a cube from a square, we need to fit a folding net of the cube inside the square.

There are a finite number of cube nets, i.e. connected figure made of 6 squares which can be folded into a cube; the following diagram shows all possibilities. So we just pick the best net, and it's clear that we should pick the shortest diameter. The best way to fit this pattern on a square is as in the following diagram:



The outer square has a side length of $4 \times \frac{2}{\sqrt{2}}$, and hence an area of $\boxed{32}$.

A4. [3 points] We can see an initial solution of $(x, y) = (24, 0)$ as $15 \times 24 = 360$. Since x is non-negative, it therefore can only take the values $0, \ldots, 24$, so $0 \leq p \leq 24$. It would be feasible to proceed by trial and error from here, but let's try to reason further.

From the solution $(24,0)$, and supposing other solutions exist, we can find the "previous solution" in the following way: if we find the "smallest" non-zero integers (x_0, y_0) for which $15x_0 + py_0 = 0$ $(*)$, then

$$(15x + py) + (15x_0 + py_0) = 360 + 0$$

Rearranged, this equals

$$15(x + x_0) + p(y + y_0) = 360$$

hence generating a new solution. Rearranging $(*)$, we get $y_0 = -\frac{15}{p}x_0$.

p is either a factor of 360 or it isn't. If it isn't, then $-\frac{15}{p}x_0$ is an integer only if x_0 is a multiple of p, so the smallest non-zero value of x_0 *is* $x_0 = p$, which gives $y_0 = 15$. This means that solutions are of the form $(24 - kp, 15k)$ for non-negative k (increasing k is repeatedly adding (x_0, y_0) our initial solution). But the smallest value of p which isn't a factor of 360 is 7, which means we can add y_0 at most $\frac{360}{7\times 15} \approx 3.4$ times, which means there are at most 4 distinct values of y, so we definitely don't have 7 solutions, let alone p.

So p – if it exists! – is a factor of 360. Therefore $15x$ always must have a factor of p, as $360 - py$ does, which limits our options of x to $1 + 360/\operatorname{lcm}\{p, 15\}$. We check when $1 + 360/\operatorname{lcm}\{p, 15\} = p$ across the possible values of p (factors of 360); this is precisely when $p = 9$. Thus the sum of values of possible p is $\boxed{9}$.

A5. [4 points] We can count the number of distinct elements by counting the number of fractions in lowest form (any fraction not in lowest form will have its lowest form also present in the set S).

By fixing $a = 1, 2, ..., 14$, we find the number of lowest form a/b as $13 + 6 + 8 + 5 + 8 + 3 + 6 + 3 + 4 + 2 + 3 + 1 + 1 = \boxed{63}$.

A6. [7 points] Let (positive) integers x, y, z be the three side lengths of the cuboid in cm. The surface area is equal to the sum of six rectangles, and hence equals $2(xy + yz + zx)$. The volume is equal to xyz.

We are given that the surface area (measured in cm^2) and volume (measured in cm^3) are both equal to some number N; so we set up an equation

$$2(xy + yz + zx) = xyz$$

and try to solve for x, y, z.

Now dividing both sides by xyz we obtain the equation

$$\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Now WLOG $x \leq y \leq z$. If $x > 6$ then we have $\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{3}{6}$, a contradiction. So we must have $x \leq 6$. But note we also must have $x > 2$ as both y and z are positive integers, so we have 4 cases for x to solve for.

Now

$$2(xy + yz + zx) = xyz$$

Gives

$$2y + \frac{(2-x)yz}{x} + 2z = 0$$

Hence

$$y\left(2 + \frac{(2-x)z}{x}\right) + 2z + \frac{4x}{(2-x)} = \frac{4x}{(2-x)}$$
$$\left(y + \frac{2x}{(2-x)}\right)\left(2 + \frac{(2-x)z}{x}\right) = \frac{4x}{(2-x)}$$
$$\left(y + \frac{2x}{(2-x)}\right)\left(z + \frac{2x}{(2-x)}\right) = \frac{4x^2}{(2-x)^2}$$

If $x = 3$, we have

$$(y-6)(z-6) = 36$$

giving us the solutions $(x, y, z) =$ (3,7,42), (3,8,24), (3,9,18), (3,10,15), (3,12,12).

If $x = 4$, we have

$$(y-4)(z-4) = 16$$

giving us the solutions $(x, y, z) = (4, 5, 20), (4, 6, 12), (4, 8, 8)$.

If $x = 5$, we have

$$\left(y - \frac{10}{3}\right)\left(z - \frac{10}{3}\right) = \frac{100}{9}$$

giving us the solutions $(x, y, z) = (5, 5, 10)$.

And finally if $x = 6$, we have

$$(y-3)(z-3) = 9$$

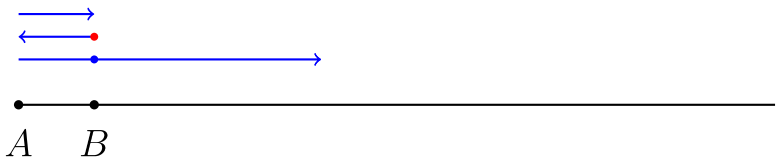
giving us the solutions $(x, y, z) = (6, 6, 6)$.

Solving for N, we find the maximum value N is $\boxed{882}$.

A7. [10 points] It's easier to start with less litres of fuel to try and spot a pattern. Suppose at A there is a reserve of n litres, and d_n is the furthest distance Jason's car can travel with those n litres. Clearly $d_0 = 0$ and $d_1 = 1$.

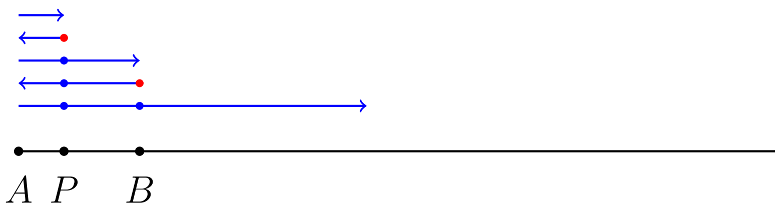
What about d_2? Jason only makes two trips, so it must be that in the first trip he's placing a reserve ahead of A with as much fuel as he can take for the second trip – call this point B and say it's at a distance x from A. B should have no more than x litres of fuel, otherwise when Jason's car reaches

B on his second trip he can only take up to x litres. Therefore, on Jason's first trip, he travels x m to get to B, stores x fuel at B, and travels another x m to return to A; so $x = \frac{1}{3}$, and hence $d_2 = 1 + \frac{1}{3}$.



The red dots represent the car leaving fuel, and the blue dots represent the car picking up fuel.

How might things work for d_3? If we place $\frac{1}{3}$ of a litre $\frac{1}{3}$ m away, then to go any further we would want to place fuel beyond B; but to do so we would need more fuel to get to some-place beyond B, which would mean taking fuel from B; but then B won't have $\frac{1}{3}$ of a litre of fuel on the last journey. What we might want to do instead is push B further along, so let's try to do just that. In the first journey, set up a point P with $\frac{3}{5}$ litres of fuel – we can put P at a distance of $\frac{1}{5}$ m from A. On the second journey, use $\frac{2}{5}$ litres to place B a distance of $\frac{1}{3}$ m from P. On the third journey, propel the car as far as you can to get $d_3 = 1 + \frac{1}{3} + \frac{1}{5}$.



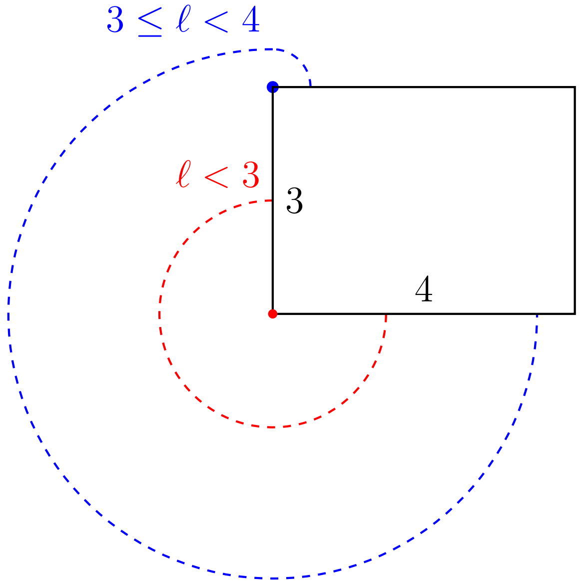
The red dots represent the car leaving fuel, and the blue dots represent the car picking up fuel.

From here there is a pattern that can be spotted – namely, that $d_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$ (showing this is quite hard!). As a simplified fraction, $d_4 = 176/105$, so $a + b = \boxed{281}$.

Column B

B1. [7 points] It suffices to vary P across a sixth of the triangle ABC (e.g. between A, the midpoint of AB — call it X — and the centroid — call it G) by symmetry of ABC. Let Z_p be the area described by the point p. Construct the vectors $\mathbf{a} = AX$ and $\mathbf{b} = AG$. For the points across $\lambda\mathbf{a}$, $\lambda\mathbf{b}$ with $0 \leq \lambda \leq 1$, the area is extreme for $\lambda = 0$ or $\lambda = 1$ (between, clearly maxima and minima aren't achieved). Anywhere between these points are also not extreme. One can also verify that $Z_A < Z_X < Z_G$, so $m = Z_A$ and $M = Z_G$. Moreover m is a third of the area of the triangle, and M is two thirds, so $\frac{M}{m} = \boxed{2}$.

B2. [4 points] It's useful to split the goat's roaming area into cases. If $\ell \leq 3$, the roaming area is just $\frac{3}{4}\pi\ell^2 \leq \frac{27}{4}\pi$ – far too small. Something interesting happens for $\ell > 3$: first, if $3 < \ell \leq 4$, the goat can wrap around the shorter side of the shed to get some extra area. In the diagram below, the dotted red line represents the boundary for $\ell < 3$, and the blue represents for some $3 \leq \ell < 4$.



With the blue boundary, the goat can travel on the $\frac{3}{4}\pi\ell^2$ as before, but also gets an extra $\frac{1}{4}\pi(\ell-3)^2$ (you can think of the goat trying to travel around the corner, and the rope is effectively tied to the blue dot with 3 units of length less). The area encompassed by the blue boundary is $\frac{3}{4}\pi\ell^2 + \frac{1}{4}\pi(\ell-3)^2$. The maximum area in this case is $\frac{49}{4}\pi$ (taking $\ell = 4$) which isn't enough still.

If $4 \leq \ell < 7$, with similar reasoning to the blue boundary the area the goat can graze in becomes $\frac{3}{4}\pi\ell^2 + \frac{1}{4}\pi(\ell-3)^2 + \frac{1}{4}\pi(\ell-4)^2$, with a maximum of 43π (with $\ell = 7$), which is enough! Set this formula equal to 28π. Then:

$$\begin{aligned}
\tfrac{3}{4}\pi\ell^2 + \tfrac{1}{4}\pi(\ell-3)^2 + \tfrac{1}{4}\pi(\ell-4)^2 &= 28\pi \\
3\ell^2 + (\ell-3)^2 + (\ell-4)^2 &= 112 \\
5\ell^2 - 14\ell - 87 &= 0 \\
(5\ell - 29)(\ell + 3) &= 0
\end{aligned}$$

Since $\ell > 0$, $\ell = 29/5$, so our final answer is $\boxed{34}$.

B3. [3 points] P_1 is clearly lying, and P_{99} is clearly telling the truth. If P_2 is telling the truth, then P_3 is lying (by P_2's statement being true); so P_4 is lying (by P_3's statement being false); so P_5 is telling the truth (by P_4's statement being false); and so on. If P_2 is telling the truth, then the truth-tellers are $P_2, P_5, P_8, \ldots, P_{3n-1}, \ldots, P_{98}$. However this list doesn't include P_{99}, so P_2 must be lying. It follows that $P_3, P_6, P_9, \ldots, P_{3n}, \ldots, P_{99}$ are the truth-tellers, hence there are $\boxed{66}$ liars.

B4. [2 points] $\triangle ADC$ has half the height of $\triangle ABC$ (considering AC as the base) and hence an area of 12. $\triangle EDC$ has its height halved again (considering DC as its base) and hence an area of 6. Again, $\triangle CME$ has half the height of $\triangle CDE = \triangle EDC$ (considering CE as the base) and hence an area of $\boxed{3}$.

B5. [3 points] Notice that f is shrike precisely when the following is true: for each x either $f(x) = x$ or $f(f(x)) = x \neq f(x)$. We count the number of shrike functions by considering these two cases.

It will clarify our logic by introducing some notation: let a_n be the number of shrike functions on the set $\{1, 2, \ldots, n\}$. The original problem asks us to find a_5, but we will derive a recurrence relation.

First suppose $f(1) = 1$. This doesn't restrict the values for $2 \leq x \leq n$, so this case gives us a_{n-1} functions.

If on the contrary we have $f(1) = y \neq 1$ (of which there are $n-1$ choices for y), then we must have $f(y) = f(f(1)) = 1$. These two elements are fixed, but everything else is free. This case gives us $(n-1)a_{n-2}$ functions.

Since these are the only possible cases, we get the identity $a_n = a_{n-1} + (n-1)a_{n-2}$ for all n. To obtain a_5, we can work backwards from smaller values of n. Check that $a_1 = 1$ and $a_2 = 2$; then we get $a_3 = 2 + 2 \cdot 1 = 4,\ a_4 = 4 + 3 \cdot 2 = 10,\ a_5 = 10 + 4 \cdot 4 = \boxed{26}$.

B6. [4 points] Filled in, the grid looks like

$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
$\frac{1}{8}$		$\frac{1}{24}$	$\frac{1}{12}$
	$\frac{7}{12}$	?	$\frac{7}{12}$

So ? $= \frac{1}{3}$, so its reciprocal is $\boxed{3}$. To get started on filling in the grid, it's helpful to note that $1/8$ must be the sum of $1/12$ and $1/24$ (all the other fractions are too big); this can then help with the right-hand $7/12$ column, which in turn helps narrows down the possibilities for the left-hand $7/12$ column.

B7. [7 points] Let's try to spot a pattern. We can say E_1 adds $\frac{1}{2}$. E_2 adds $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, and E_3 adds $\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$. This leads us to form the equation

$$\frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{9900}$$

After some algebra, we get $n^2 + n - 9900 = 0$, which factorises to $(n - 99)(n + 100) = 0$, so $n = \boxed{99}$.

Alternatively, you could've noticed that $2, 6, 12, \ldots$ are twice the triangle numbers $1, 3, 6, \ldots$, so equivalently we want to know which double triangle number 9900 is. That'll form the same equation as above.

Column C

C1. [4 points] First we solve for each angle of $\triangle ABC$: Draw the lines AB, BC, CA. Notice that the lines AB, T_D, T_F, AC bound a quadrilateral Q_A.

Since T_D is the tangent line to circles X and Y, it must be perpendicular to the radius (vector) of each circle, i.e. T_D is perpendicular to AD and DB. Therefore the angles of Q_A at D and F are right angles.

Recall that angles in a quadrilateral sum to 360°, which gives us $\angle BAC = 360° - 90° - 90° - \angle(T_D, T_F) = 30°$.

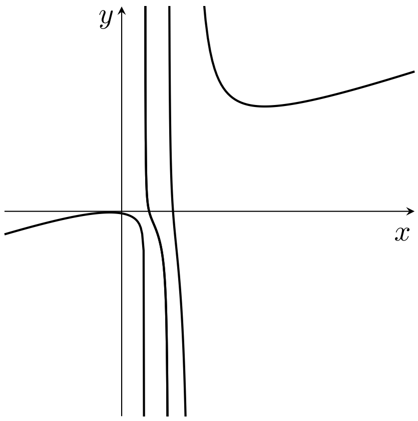
Similarly we obtain $\angle ABC = 90°$ and $\angle BCA = 60°$. (TIP: check that these three angles really add to 180°!)

Now that we have all angles, use the sine formula:

$$\begin{aligned} \frac{AB}{\sin \angle BCA} &= \frac{BC}{\sin \angle BAC} \\ \frac{5}{\sqrt{3}/2} &= \frac{BC}{1/2} \\ BC &= \frac{5\sqrt{3}}{3}. \end{aligned}$$

It follows that $10a + b = \boxed{53}$.

C2. [3 points] The expression goes to ∞ as $x \to \infty$, and to $-\infty$ as $x \to -\infty$. Further, there are vertical asymptotes at $a, 2a, 3a$, which are all distinct since $a \neq 0$. This divides the plane into $\boxed{5}$ regions as can be seen in a graph.



C3. [2 points] For brevity, denote $S(n)$ denote the sum of the digits of n. We can write $2S(n) = S(3n)$. Note $S(3n)$ is a multiple of 3, so $S(n)$ is a multiple of 3, so n is a multiple of 3, so $S(3n)$ is a multiple of 9, so n is a multiple of 9. With this insight, our search of choices is reduced greatly, so only the multiples of 9 needing to be checked. We can then see that 63 is the only such number with this property, so the answer is $\boxed{63}$.

C4. [1 point] Let's go angle-chasing! Say $\angle ABC = x^\circ$. Since CPB is isosceles, $\angle PCB = x^\circ$ so $\angle BPC = 180 - 2x^\circ$. Since the angles around P add to 180°, $\angle APC = 2x^\circ$. Since ACP is isosceles, $\angle PAC = 2x^\circ$, so $\angle ACP = 180 - 4x^\circ$. Now $\angle ACB = \angle ACP + \angle BCP = 180 - 3x$. Since ABC is isosceles, $\angle ACB = \angle CAB$ so $180 - 3x = 2x$, which solves to $x = \boxed{36^\circ}$.

C5. [2 points] Let's set up our equation:

$$\frac{2}{\sqrt[3]{3} - 1} = \sqrt[3]{a} + \sqrt[3]{b} + c$$

Multiplying up,

$$\begin{aligned} 2 &= (\sqrt[3]{3} - 1)(\sqrt[3]{a} + \sqrt[3]{b} + c) \\ \implies 2 &= \sqrt[3]{3a} + \sqrt[3]{3b} + c\sqrt[3]{3} - \sqrt[3]{a} - \sqrt[3]{b} - c \end{aligned}$$

This looks a bit icky, but we can work with this. Importantly, c is positive so $-c$ is negative; for the right-hand side to be equal to a positive integer, at least one of the surds must be an integer! It won't be $c\sqrt[3]{3}$, which leaves either $\sqrt[3]{3a}$ or $\sqrt[3]{a}$ (or b, but it doesn't really matter which we focus on).

We should try small values of a which give us integers; our candidates are either $a = 8$ (since $\sqrt[3]{a}$ would equal 2) or $a = 9$ (since $\sqrt[3]{3a}$ would equal 3). One can spot that $a = 9$ leads to $b = 3$ (which cancels $\sqrt[3]{3b}$ with $\sqrt[3]{a}$) and $c = 1$, so $a + b + c = \boxed{13}$.

C6. [3 points] The prime factorisation of 12! is $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$.

For each factor of 12!, each prime factor can be present with powers ranging from 0 to its power in the above factorization. For example, the factor 5 can be present as $5^0, 5^1$ or 5^2 – thus it provides 3 choices. Each prime number offers such choices independently, so we should multiply the number of choices together by the product rule of counting (but we should ignore choices of 2 since we only want odd factors). Thus, the number of odd factors is $6 \cdot 3 \cdot 2 \cdot 2 = \boxed{72}$.

C7. [4 points] Without loss of generality, suppose $a \leq b \leq c$. We can spot the solutions $(a, b, c) = (2, 3, 6)$ and $(2, 4, 4)$. Suppose for a contradiction that $a = 2$ and $b > 4$; then $\frac{1}{c} = 1 - \frac{1}{a} - \frac{1}{b} > \frac{1}{2} - \frac{1}{4}$. So $\frac{1}{c} > 1/4$ so $c < 4$: a contradiction (by ordering of b, c). If $a = 3$, we may also spot $(a, b, c) = (3, 3, 3)$. One can make a similar argument that, if $a = 3$ and $b > 3$, we get no well-ordered solutions. Lastly, if $a > 3$ then since $a \leq b \leq c$ the sum can only be as big as $\frac{3}{4}$, which is not big enough. Now, removing the well-ordering restriction we get $\boxed{10}$ solutions.

Column D

D1. [3 points] Notice that the max is $N = 15$, achieved when no three points are colinear. When you have 3 colinear points, you collapse 3 lines into 1, reducing N by 2. When you have 4 colinear points, N reduces by 5, 5 colinear points reduces by 9, 6 colinear points reduce by 14. Considering which combinations of colinear points we can have gives us an easy way to find all possible N. Such possible values for N are 15, 13, 11, 10, 9, 8, 7, 6, 1. Giving us the answer $\boxed{80}$.

D2. [2 points] Let θ be the angle between the sides with length 20 and 25 and c be the final side length of the triangle (opposite of θ). Then by the cosine rule, $c = \sqrt{20^2 + 25^2 - 2 \cdot 20 \cdot 25 \cos(\theta)} = \sqrt{1025 - 1000 \cos(\theta)}$. Now $-1 < \cos(\theta) < 1$ as $0° < \theta < 180°$ so we know that every value of $5 < c < 45$ is achieved. As we need c to be an integer, the number of possible values of c is $\boxed{39}$.

D3. [1 point] If we try expanding with the smaller powers first, note

$$(1 + x)(1 + x^2) = 1 + x + x^2 + x^3$$

and

$$(1 + x + x^2 + x^3)(1 + x^4) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$$

This (correctly) indicates that the expression is

$$1 + x + x^2 + \cdots + x^{63}$$

which means the sum of the coefficients is $\boxed{64}$ (one for each power of x). Similarly to 2. (e) of Lock-in, we can reason this expansion without brute force by thinking about the terms in each bracket I multiply, and which ones I can multiply to get each power from x^0 to x^{63}.

D4. [0 points] Free space! You don't need to submit anything.

D5. [1 point] Since there are $\frac{10 \cdot 11}{2} = 55$ elements, the median is the 28th element in ascending order, which turns out to be $\boxed{7}$.

D6. [2 points] Suppose for a contradiction that $n \leq 5$ and $n \times AB = ACB$. Then we know that

$$100A \leq ACB = n \times AB \leq 5 \times AB = 50A + 5B$$

But this means that $10A \leq B$, which is a contradiction as A is non-zero and $0 \leq B \leq 9$. $n = 6$ is possible as $n \times 18 = 108$. So the smallest n is $\boxed{6}$.

D7. [3 points] If we let θ be the angle between the two strips, then by the sine area rule, we get $2 \cdot 1 \cdot \sin\theta = 4$. Thus $\theta = \boxed{30^\circ}$.

Column E

E1. [4 points] By the pigeonhole principle, there must be one of a_i which is odd, and thus $a_i^{a_i} - 1$ is even. So for the last digit of the product to be 0, we need one of a_i to be such that $(a_i^{a_i} - 1)$ has a factor of 5. The only a_i that satisfy this are $a_i = 1, 4, 6, 8$ thus the only way to NOT have the last digit of the product be 0 is if $(a_1, a_2, a_3, a_4, a_5)$ is some permutation of $(2, 3, 5, 7, 9)$. Thus $p = \frac{1}{126}$ so the answer is $\boxed{126}$.

E2. [3 points] Writing out the first few terms of the sequence:

$$a, b, a+b, -a, b, -a-b, -a, -b, -a-b, a, -b, a+b, a, b$$

Since the a, b at the end falls on $n = 13, 14$ respectively, the pattern continues. One can see that the 97th and 98th term are a, b, and hence the 100th and 101st term are $-a, b$. Thus $100 - a/b = \boxed{101}$.

E3. [2 points] Let Q' denote the reflection of the point Q over T. Then we know that the area of QSQ' is $\frac{4}{9}$ the area of PQR. But we also know that QSQ' is similar to QPR and thus the side lengths of QSQ' are $\sqrt{\frac{4}{9}} = \frac{2}{3}$ the side lengths of QPR. And thus as the perimeter of $QSQ' = 90$, the perimeter of PQR is $90 \cdot \frac{3}{2} = \boxed{135}$.

E4. [1 point] $2+\sqrt{20+\sqrt{202+\sqrt{2025}}}=2+\sqrt{20+\sqrt{247}}$. We have the bound $15<\sqrt{247}<16$ so $2+\sqrt{35}<2+\sqrt{20+\sqrt{247}}<2+\sqrt{36}<8$. Now $5.5^2=30.25$ so $\sqrt{35}>5.5$ thus $7.5<\sqrt{20+\sqrt{247}}<8$ so the value rounded to the nearest integer is $\boxed{8}$.

E5. [2 points] The circle has centre at $(5,12)$. So radius of the circle is

$$\sqrt{(8-5)^2+(16-12)^2}=5$$

The distance from the centre of the circle to the origin is $\sqrt{12^2+5^2}=13$, thus the shortest distance from the origin to a point on the circle is $13-5=\boxed{8}$.

E6. [3 points] A stretch by b implies that the new function is $f(x)=b\ln x$. Taking natural logarithms on our equation,

$$\begin{aligned} x^2=8^{f(x)} &\implies \ln\left(x^2\right)=\ln\left(8^{f(x)}\right) \\ &\implies 2\ln x=f(x)\ln 8 \\ &\implies \frac{2\ln x}{\ln 8}=b\ln x \\ &\implies b=\frac{2}{\ln 8} \end{aligned}$$

Thus $e^{2/b}=e^{\ln 8}=\boxed{8}$.

E7. [4 points] One way to construct such a group of people is to have a "universal friend" who is friends with everybody. The other 98 people pair up to form a triangle with this universal friend. It turns out that this is the only way to construct such a group, but proving this is very difficult! The universal friend has friendship number 98, and the other 98 people have friendship number 2. This gives $\boxed{294}$.

Column F

F1. [7 points] Consider the Binomial expansion of $(10+1)^{2025}$. All of the terms with 10^4 or higher are essentially irrelevant in how they affect the last four digits, so we only need to consider the last four digits of

$$^{2025}C_3 10^3 1^{2022}+{}^{2025}C_2 10^2 1^{2023}+{}^{2025}C_1 10^1 1^{2024}+1$$

We know $^{2025}C_1=2025$, we can calculate

$$^{2025}C_2=\frac{2025!}{2!2023!}=\frac{2025\times 2024}{2}=2025\times 1012=2049300$$

In fact, it suffices to calculate 25×12 because

$$2025 \times 1012 = (2000 + 25)(1000 + 12)$$

and the terms with factors of 1000 don't affect the last four digits (since we're multiplying this term by 10^2). For ${}^{2025}C_3$, as long as we know there's a factor of 100, the 10^3 means this has no effect. Indeed,

$${}^{2025}C_3 = \frac{2025!}{3!2022!} = \frac{2025 \times 2024 \times 2023}{6} = 675 \times 1012 \times 2023$$

675 has a factor of 25, and 1012 has a factor of 4, so we have a factor of 100.

Therefore, the only terms affecting the last four digits is 1 and 20250, so the last four digits are $\boxed{0251}$.

F2. [4 points] The sum of digits of any number divisible by 9, is also divisible by 9. Since 3^{2025} is divisible by 9, so will its sum of digits (the new number), and hence so will also the third number. Since the sum of a number bigger than 10 is less than the number, this continues till we reach a single digit, and hence the answer is $\boxed{9}$.

F3. [3 points] If Angie and Rosie had both the even numbers on their backs, then Ella would've known that her number was odd. Thus at least one of Angie and Rosie has an odd number on their back.

Similarly, if Rosie had an even number on her back, then Angie would've known that she had an odd number on her back, so Rosie must have an odd number on her back.

Thus the sum of possible numbers $= 1 + 3 + 2025 = \boxed{2029}$.

F4. [2 points] Note $f(0°) = -50$ and $f(90°) = 50$. For any other values, either $|\sin x| < 1$, in which case the subsequent sine terms become negligible, or $|\cos x| < 1$, in which case the subsequent cosine terms become negligible. There is therefore no way for $|f(x)| > 50$; so $M - m = \boxed{100}$.

F5. [3 points] Taking the reciprocal, we get

$$\frac{1}{\sqrt{n+1} - \sqrt{n}} > 24$$

Can we rationalise this? If we multiply top and bottom of the left-hand side by $(\sqrt{n+1} + \sqrt{n})$, we get

$$\sqrt{n+1} + \sqrt{n} > 24$$

Now $\sqrt{144} + \sqrt{144} = 24$, so $\sqrt{144} + \sqrt{143} < 24$ and $\sqrt{145} + \sqrt{144} > 24$. Therefore the smallest binky number is $\boxed{144}$.

F6. [4 points] There's a naïve approach, where we take the biggest two numbers at each available opportunity. This gives

$$(5,4) \mapsto 33; \quad (33,3) \mapsto 69; \quad (69,2) \mapsto 140; \quad (140,1) \mapsto 281.$$

But we can do better by trying to take advantage of the $a + 7b$ option. If we get two medium-sized numbers then $a + 7b$ can be quite big! Indeed, the approach

$$(3,4) \mapsto 25; \quad (2,5) \mapsto 19; \quad (19,25) \mapsto 158; \quad (158,1) \mapsto 317$$

is optimal, so the largest number you can create is $\boxed{317}$.

F7. [7 points] For a circle with $a \leq 2$, Player 67 automatically wins, since no point is more than half the circumference ($\leq 2\pi$) away from the first point placed.

For $a > 2$, Player 456 can always place points diametrically opposite to the points placed by Player 67 in the previous turn. By symmetry, if Player 67's point is valid, Player 456's point must be. This continues until Player 67 loses.

Thus the answer is $2 \cdot 456 = \boxed{912}$.

Column G

G1. [10 points] Note that if 9 divides n, then 9 divides the sum of its digits $d(n)$. To show this, suppose that the ones digit of n is a_0, its tens digit is a_1, and so on, so that $n = a_0 + 10a_1 + ... + 10^k a_k$. Then $d(n) = a_0 + a_1 + ... + a_k$, so $n - d(n) = (1-1)a_0 + (10-1)a_1 + ... + (10^k - 1)a_k$. Since each of these terms are a multiple of 9, their sum must be a multiple of 9 too. But if 9 divides n and 9 also divides $n - d(n)$, then it must necessarily be true that 9 divides $d(n)$ as well.

Now, since 9 divides 2025 and 2025 divides n, 9 divides n. Thus, 9 divides $d(n)$ and by extension 9 also divides $d(d(n))$. Thus, we know that $2025 \cdot 9 \cdot 9$ divides n.

Directly checking, we note that $n = 2025 \cdot 9 \cdot 9 \cdot 2$ or $n = 2025 \cdot 9 \cdot 9 \cdot 3$ each case giving n a total of $\boxed{11}$ prime factors.

G2. [7 points] The roots of this equation are $1 \pm \frac{\sqrt{2}}{2}$. Importantly, $2 > \alpha > 1 > \beta > 0$, and $\alpha + \beta = 2$. Considering decimal expansions, let $\alpha = 0.a_1 a_2 a_3 \ldots$ and $\beta = b_1 b_2 b_3 \ldots$. Then

$$\alpha + \beta = 0.a_1 a_2 a_3 \cdots + 0.b_1 b_2 b_3 \cdots = 1.999\ldots$$

so for each i, $a_i + b_i = 9$. Given that $a_{2025} = 2$ (which is true, I checked!), it follows that $b_{2025} = \boxed{7}$.

G3. [4 points] To solve this problem, we count the highest power of 2 that divides 25!. This is given by

$$\left\lfloor \frac{25}{2} \right\rfloor + \left\lfloor \frac{25}{4} \right\rfloor + \left\lfloor \frac{25}{8} \right\rfloor + \left\lfloor \frac{25}{16} \right\rfloor = 22$$

where $\lfloor x \rfloor$ is x rounded down to the nearest integer. We also count the highest power of 5 that divides 25! which is given by

$$\left\lfloor \frac{25}{5} \right\rfloor + \left\lfloor \frac{25}{25} \right\rfloor = 6$$

To get a zero at the end of a number, we need a factor of $10 = 2 \cdot 5$; so the highest possible number of zero digits at the end of $25! \cdot 5^n$ is 22. As we already have six 5's in 25!, the least value of n is $\boxed{16}$.

G4. [3 points] One can spot that $7 + 4\sqrt{3} = 4 + 2 \cdot 2\sqrt{3} + 3 = (2 + \sqrt{3})^2$, so in fact $z = 2$. It follows that $z^4 = \boxed{16}$.

G5. [4 points] Considering the capital's remainder when divided by 4, after either action of **A** or **B** it always decreases by one. Thus one can only break even on games of lengths which are multiples of 4. But which such games? It is not too hard to check that one can break even on a game of length 4 (e.g., **BAAA**) so we can break even on any game length which is a multiple of 4 less than 50 (by repeating **BAAA**, for example) so we should add $4 + 8 + 12 + \cdots + 48$. This gives $\boxed{312}$.

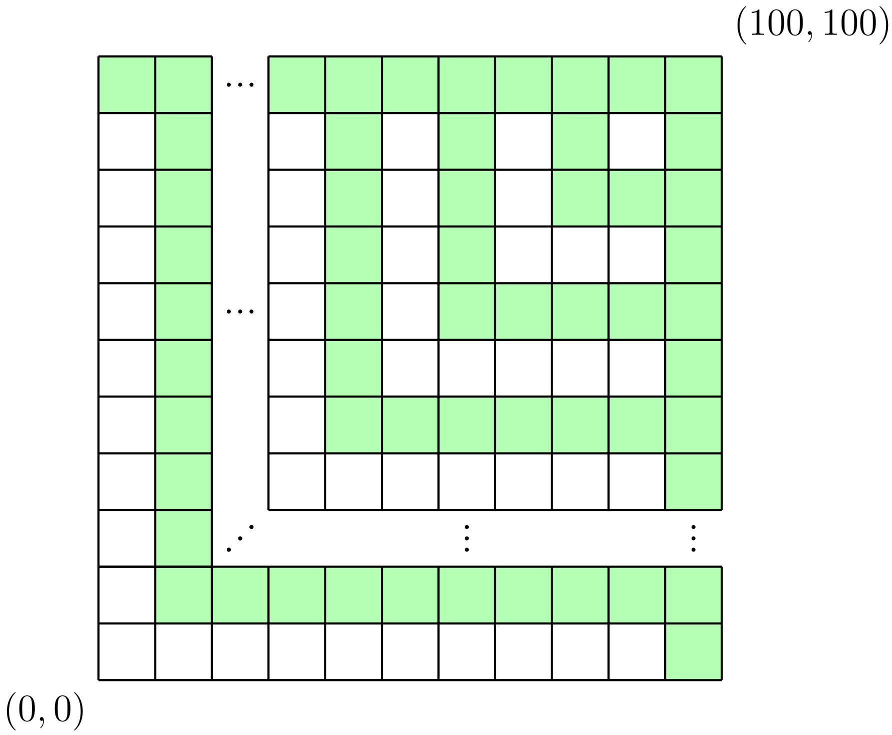
G6. [7 points] If 2024 is divisible by $(n + 1)$, then whatever is added by Healy, Dobby can always add a number so that the total is a multiple of $n + 1$. At the end, Dobby will reach 2024, forcing Healy to lose.

For other n, Healy can start with a number such that, effectively, the game is restarted with a modified total a to reach, such that $a - 1$ is divisible by $n + 1$, with Dobby starting. Utilising the same strategy as above, Healy can win.

For example, if $n = 6$, Healy can add 1 so that the game becomes a 'Game of 2024', with Dobby starting, and $2024 - 1 = 2023$ is divisible by $6 + 1 = 7$. Then whatever Dobby plays, Healy can make the total a multiple of 7. At the end, Healy will reach 2023 of the modified game, which is 2024 of the original game, forcing Dobby to lose.

Thus the only n for which Dobby has a winning strategy are those such that $n + 1$ divides 2024. $n = 1, 3, 7$ are the only such n in the set. Hence, the answer is $\boxed{3}$.

G7. [10 points] The winning spaces are shaded below:



It's easier to calculate that the losing area is $1 + 5 + 9 + \cdots + 197 = 4950$, and hence the winning area is $100^2 - 4950 = \boxed{5050}$.