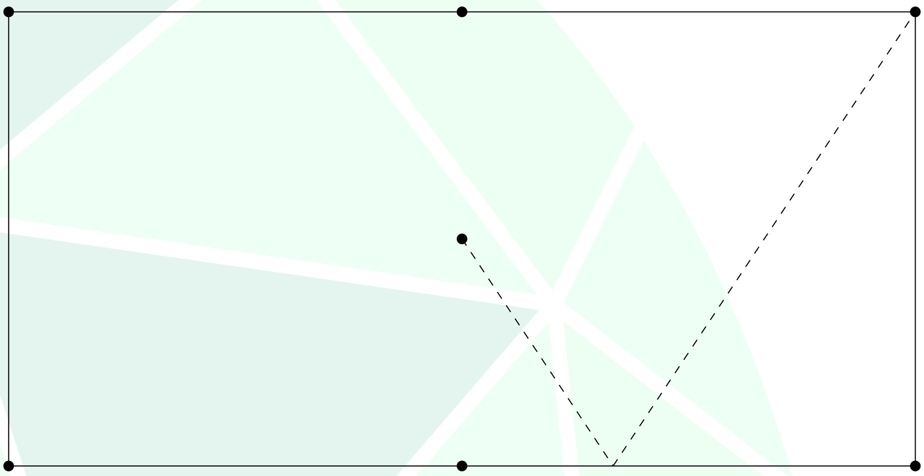
the invariants

puzzles & pizza, tt25 week 1

hit and pray

Consider a standard rectangular billiards table with a ball placed exactly at its centre. Suppose we model both the ball and the pockets as points (ie. the ball can only be sunk if it lands exactly into a pocket).



[Above shows a possible trajectory to sink the ball after 1 bounce.]

Assuming all collisions are elastic, how many ways are there to strike the ball so that it lands in a pocket after exactly 2025 bounces?

Can you generalize this to n bounces?

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i'm stoked about this

1. Let P be a polygon in the xy-plane, with vertices $p_i = (x_i, y_i), 1 \leq i \leq n$, and edges $p_i \to p_{i+1}$, that are positively oriented (traversing them happens in an anticlockwise sense).
Let $\boldsymbol{F} = -\frac{y}{2}\boldsymbol{i} + \frac{x}{2}\boldsymbol{j}$

(a) Using Stokes' Theorem, deduce 'the Shoelace Formula' for the area of P:

$$A(P) = \frac{1}{2}\sum_{i=1}^{n-1}(x_i y_{i+1} - x_{i+1} y_i)$$

[Hint: $A(P) = \iint_P \nabla \wedge \boldsymbol{F} \cdot \boldsymbol{dS}$]

2. Now consider a polyhedron D in $\mathbb{R}^3$, with faces $F_1, \ldots, F_n$.
The centroid of any polygon P is defined to be the point $\boldsymbol{c} = \frac{1}{A}\iint_P \boldsymbol{r}dxdydz$

(a) Using a suitable vector field $\boldsymbol{F}$, and applying the Divergence Theorem, show that the volume of D is given by

$$V(D) = \frac{1}{3}\sum_{i=1}^{n}(\boldsymbol{c}_i \cdot \boldsymbol{n}_i)A_i$$

where $\boldsymbol{c}_i, \boldsymbol{n}_i, A_i$ are the centroid, unit normal (pointing away from D), and area of face F_i respectively.

[Hint: on a face F_i, the unit normal is constant]

(b) This formula can be generalized to $V(D) = \frac{1}{3}\sum_{i=1}^{n}(\boldsymbol{p}_i \cdot \boldsymbol{n}_i)A_i$, where $\boldsymbol{p}_i$ is ANY point on F_i. Can you think why?

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packing balls

Suppose we have a cuboid that can fit 100 spherical balls of radius 1.

Is it always possible for this same cuboid to fit 800 spherical balls of diameter 1?

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probability

(a) You are given 2 independent random numbers from the uniform distribution on $[0, 1]$, revealed one at a time over 2 rounds. As each number is revealed, you must immediately label it as either "top" (larger) or "bottom" (smaller). You cannot change your labels after assigning them.

You win if both labels correctly match the final ordering of the two numbers. What is the maximum probability of winning, using the optimal strategy?

(b) This time, you are given 3 independent random numbers from the uniform distribution on $[0, 1]$, revealed one at a time over 2 rounds. As each number is revealed, you can either label it as "top", "middle", or "bottom".

What is the maximum probability this time?

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true or false

Which of the following claims are true or false? Try to prove or find a counterexample for each!

(a) If $G_1 \cong G_2$, $H_i \trianglelefteq G_i$ for $i = 1, 2$, and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

(b) Let $H \leq G$. Then $H \trianglelefteq G$ iff H is a union of conjugacy classes in G.

(c) If G is Abelian, then every subgroup of G is a normal subgroup.

(d) If every subgroup of G is a normal subgroup, then G is Abelian.

[Note: $H \trianglelefteq G$ if H is a normal subgroup of G]